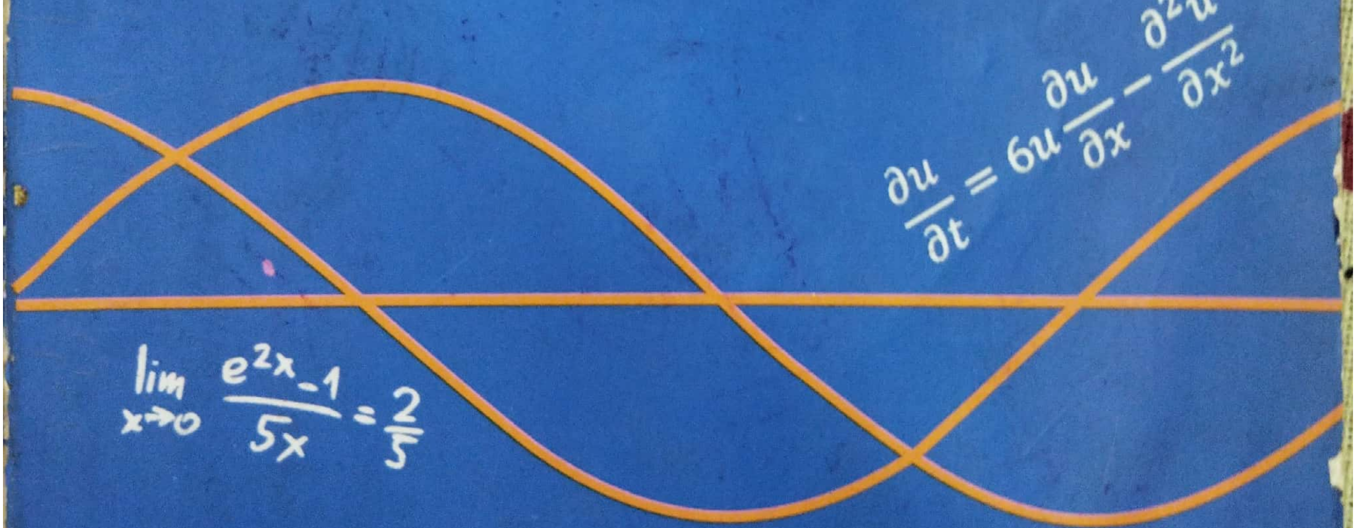
Revised Edition 2016

Differential Calculus

Dr. Gorakh Prasad


$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}$
$\lim_{x \to 0} \frac{e^{2x}-1}{5x} = \frac{2}{5}$

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Text Book

on

DIFFERENTIAL CALCULUS

[For B.A. & B.Sc. Classes]

BY

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REVISED BY

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Preface to the First Edition of Differential Calculus by Dr. Gorakh Prasad

This book has been written to meet the requirements of the B.A. and B.Sc. students of Indian Universities. The treatment of the subject is in keeping with the modern theory of functions, but is at the same time simple. The excessive refinements of the present-day higher mathematics have been deliberately excluded, as being unsuitable for the beginner, and geometrical intuition and graphical illustrations have been freely drawn upon to drive home to the student results obtained by analytical methods. With the exception of the last chapter and of a few articles here and there, the whole of the subject-matter here presented has stood the test of actual classroom use for a number of years, and the author believes that the book would be found to be in every way suitable for the B.A. and B.Sc. classes.

The concept of limits, usually found very difficult by students, is derived in Chapter I from the notion of continuity, in a way which, it is believed, will make it very simple. The great variety of applications of the Differential Calculus to various branches of knowledge, which is given in Chapter IV, immediately after the student has learnt differentiation, will, it is expected, create in him a more lively interest towards the subject than is possible when it is attempted to teach the entire principles of the Calculus first. The subject of Asymptotes has been developed as a natural extension of that of tangents and will be found perhaps easier to grasp than when treated otherwise. An attempt has been made to make the rules regarding Curve Tracing so systematic as to make this subject quite easy. The historical and biographical notes interspersed throughout the book and the short historical sketch at the end will, it is hoped, prove interesting. The historical sketch will serve also to give the student the right outlook as regards 'infinitely small quantities' and related topics.

The book contains just a little more than is necessary for the usual course, and no hesitation need be felt in omitting some of the articles. The number of exercises will also be found to be ample. Of these some are original, a large number has been taken from the examination papers of various universities, and the others are such as are common to practically all text-books on the subject. Some of the exercises have been taken from the excellent collection of Leib, entitled 'Problems in the Calculus'.

I am greatly indebted to Mr. Harish Chandra Gupta, M.Sc., a pupil of mine, who has read with great care the proofs of the whole of the book and verified most of the examples. My thanks are due also to my colleagues Dr. B. N. Prasad, Ph.D., D.Sc. and Mr. R. N. Choudhuri, B.A. (Cantab.), to my friends Mr. S. D. Seth, M.Sc., Mr. R. S. Varma, M.Sc. and Mr. R. D. Misra, M.Sc., to my pupils Mr. Lakshmi Narayan Sharma, M.Sc. and Mr. Sadanand Mukerji, M.Sc. and to some of the students of my B.Sc. class, who have all very generously helped me in reading the proofs or verifying the examples, or have offered valuable criticism and advice.

University of Allahabad GORAKH PRASAD
July 1936

PREFACE

The first edition of this book written by my grandfather, late Dr. Gorakh Prasad, came out in 1936. It was revised several times over the years by my father, Dr. Chandrika Prasad.

The current edition of the book has been revised in accordance with the latest UGC syllabus. A large number of solved examples have been added, especially frequently asked questions as also a new chapter on Jacobians.

May 2016 RASHI KANSAL

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1

Real Numbers and Functions

1.1. Related quantities. The area of a circle depends upon its radius; the velocity of a falling particle depends upon the distance through which it has fallen; the cube of a number depends upon the number itself. In these examples the radius of the circle, the distance through which a particle has fallen, and the number which is cubed are variable. They are not fixed. The quantities which depend upon them – area, velocity and cube respectively – also vary.

Differential Calculus deals with the way in which one quantity varies when the other, on which it depends, is made to vary, and with allied topics. But to get a theory which would be applicable to any pair of related quantities, we take up the study of pure numbers instead of concrete quantities; and to avoid any looseness in the arguments we shall carefully define the words we shall use. But before we do this, we introduce some basic concepts of numbers and sets.

1.2. Numbers. The numbers we generally use in elementary calculus are real numbers. In advanced mathematics complex numbers play an important role, but in this book we shall almost exclusively deal with real numbers. In future the word "number", unless specified, should be taken to mean "real number".

The simplest type of numbers are the ***natural numbers***: 1, 2, 3, 4, ...; then come the *integers*:

$$\ldots, \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots .$$

Next come the ***rational numbers***, which are ratios of integers. Every rational number has the form m / n, where m

(iv) *If* $a > b$, *then* $a + c > b + c$.

(v) *If* $a > b$ *and* c *is positive number*, $ac > bc$.

(vi) *If* $a > b$ *and* c *is a negative numbers*, $ac < bc$.

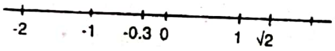
(vii) *If* $a + c > b$, *then* $a > b - c$ (transposition of a term).

A particular case of transposition is :

$$\text{If } a > b, \text{ then } -b > -a.$$

We sometimes use the notation $a \geq b$, which is an abbreviation of "$a > b$ or $a = b$". $a \leq b$ can be interpreted similarly. Again, the notation $a < x < b$ is an abbreviation of "$a < x$ and $x < b$". $a < x < b$ *means that the number* x *lies between the numbers* a *and* b.

1.22. Geometrical Representation. The real numbers can be represented geometrically by points on a line (known as the *real line*). Draw a straight line and mark on it a point 0 and another point 1. Then 01 gives the unit length and other points can be marked on the line on the same scale. Each point on the line corresponds to one real number, and vice-versa.



A relation $1 < x < 2$ will mean that the point corresponding to x lies on the line between the points marked 1 and 2.

We often call the point on the real line corresponding to the number x as the "point x"; and use the words "number x" and "point x" interchangeably.

1.23. Distance between two points. The distance between two points x and a on the real line is denoted by $|x - a|$ and defined as follows:

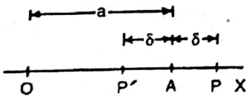
$$|x - a| = a - x \quad \textit{if} \quad x < a,$$

$$|x - a| = x - a \quad \textit{if} \quad x \geq a,$$

In other words, it is the numerical difference between the numbers x and a.

If δ be a positive number, the relation

$$|x - a| = \delta$$



means that the point x lies within a distance δ of a; i.e. x lies between $a - \delta$ and $a + \delta$. The set of points lying between $a - \delta$ and $a + \delta$ is called the δ-***neighbourhood of a.***

1.3. Absolute Value. The numerical difference $|x - a|$ is the called absolute value of $x - a$. We define the absolute value of a number as follows.

The ***absolute value*** $|a|$ of a real number a is defined by

(i) $|a| = a$ if a is positive or zero,

(ii) $|a| = -a$ if a is negative.

Thus $|2| = 2, \ |-2| = -(-2) = 2,$

and $|1.7 - 2| = |-0.3| = -(-0.3) = 0.3.$

Evidently $|a| = |-a|$, for every real number.

We give below a few simple but useful results.

(i) $|ab| = |a| . |b|, \ |a/b| = |a| / |b|$;

(ii) $|a + b| \leq |a| + |b|$;

(iii) $|x| < |c|$ *if and only if* $-c < x < c$; c *being positive.*

PROOFS (i) Follows directly from the definition.

(ii) Various cases arise. If a and b are both positive

$$|a + b| = a + b = |a| + |b|$$

If a and b are both negative

$$|a + b| = -(a + b) = -a - b = |a| + |b|$$

If a is positive, b negative and $a + b$ positive.

$$|a + b| = a + b = a - (-b) = |a| - |b| < |a| + |b|$$

If a is positive, b negative and $a + b$ negative,

$$|a+b| = -(a+b) = -a-b = -|a|+|b| < |a|+|b|$$

The results are similar when a is negative and b positive. Finally, if a or b is zero, $|a+b| = |a|+|b|$

(iii) If x is positive, $|x| = x$. So $|x| < c$, if $x < c$, and vice versa. Also $-c < x$, since x is positive.

If x is negative, $|x| = -x$. So the inequality $|x| < c$ is equivalent to

$$-x < c \quad \text{or} \quad -c < x.$$

Moreover $x < c$, since x is negative.

1.4. Sets and intervals. In mathematics the concept of a set is a fundamental one. Any collection of objects is referred to as a ***set***. Thus, we have the set of integers, the set of 3 cube roots of unity, the set of all the circles in a plane, or the set of examples given after this article. The objects in the collection are called ***elements*** of the set. The sets are often denoted by single capital letters, e.g. A, B, S or X. The elements are denoted by lower case letters $a, b, c, \ldots x, y, z$. To indicate that a is an element of the set S we use the notation $a \in S$. This is read as "a is in S" or "a belongs to S".

The constitution of a set can be indicated by displaying the elements within braces { }. For example

$$\{2, 3, 5, 7, 11, 13, 17, 19\} \qquad (1)$$

is the set of primes less than 20, all of which have been displayed in (1). Alternatively, we describe the set by displaying a typical element and stating the property satisfied by this element. For example

$$\{x \mid x \text{ is an even integer}\} \qquad (2)$$

is the set of all elements x which satisfy the property that x is an even integer. Thus it is the set of all even integers. We read (2) as the "set of all x such that x is an even integer". In this notation the set (1) may be written as $\{x \mid x = \text{prime} < 20\}$. Another example is

$$\{x \mid x > 5\} \qquad (3)$$

which is the set of all numbers greater than 5.

It is expected that the student has studied sets in Algebra and is familiar with 'union' and 'intersection' of sets.

In calculus we often use sets of real numbers lying between two particular numbers. These are called ***intervals***. Thus, if a and b are two numbers ($a < b$) the set of all real numbers lying between a and b, is called the ***open interval*** from a to b, and is denoted by (a, b). Thus

$$(a, b) = \{x \mid a < x < b\}$$

The open interval (a, b) does not contain the end points a, b. The interval which contains the end points also is denoted by $[a, b]$, and is called the ***closed interval*** from a to b. Thus

$$[a, b] = \{x \mid a \le x \le b\}$$

We can also have intervals closed at one end and open at the other, e.g.

$$[a, b) = \{x \mid a \le x \le b\}$$

$$(a, b] = \{x \mid a < x \le b\}$$

In calculus the symbol ∞, called ***infinity***, is found useful in many situations dealing with large numbers. The student should realise, however, that infinity is not a number, and is introduced in a particular situation in a particular sense. In connection with intervals, we use the notation (a, ∞) to denote the set of all real numbers greater than a. Thus

$$(a, \infty) = \{x \mid x > a\}$$

Similarly $\qquad (-\infty, a) = \{x \mid x < a\}$

The set of all real numbers may be denoted by $(-\infty, \infty)$.

1.5. Ordered Pairs. An ***ordered pair*** (a, b) is a set of two elements for which the order of the elements is of significance. Thus, $(a, b) \ne (b, a)$ unless $a = b$. In this respect (a, b) differs from the set $\{a, b\}$.

Again, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

If X and Y are two sets, then the set of all ordered pairs (x, y), such that $x \in X$ and $y \in Y$, is called the ***Cartesian product*** of X and Y. It is denoted by $X \times Y$.

A subset **R** of $X \times Y$ is called a ***relation*** of X on Y. It gives a correspondence between the elements of X and Y. If (x, y) be an element of **R**, then y is called an ***image*** of x.

A relation in which each element of X has a single image is called a ***function***. Because of its importance we give in the next section an independent definition of function.

Ex. If $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$, then

$$X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}.$$

$$R_1 = \{(1, a), (2, b), (3, c), (4, b)\}$$

is a relation as well as a function, while

$$R_2 = \{(1, a), (2, b), (2, c), (3, c)\}$$

is a relation but not a function (since 2 has two images).

1.6. Functions. Consider the equation

$$y = x^2$$

It gives a rule which determines for each number x a corresponding number y. The set of all such pairs of numbers (x, y) determines a *function.*

The rule defining a function need not be an equation, nor need we take every value of x. For example, the table

x:	1	2	3	4	5
y:	2	-1	-2	-1	2

gives for each x a corresponding y; and therefore defines the function

$$\{(1, 2), (2, -1), (3, -2), (4, -1), (5, 2)\}.$$

DEFINITION. *Let X and Y be two sets, and suppose that to each element x of X there corresponds, by some rule f, a unique element y of Y. Then the set f of all ordered pairs (x, y) is called a* ***function*** *or* ***mapping*** *from X to Y.*

We write $\qquad f : X \to Y$

to denote " the function f which maps set X into set Y" or "the function f defined by the rule $y = f(x)$".

Mapping indicates the geometrical aspect of the correspondence between the elements of X and Y, while function indicates its analytical aspect.

The set X is called the ***domain*** of the function. The element y, which corresponds to the element x, is called the ***image*** or ***value of the function at x***. It is denoted by $f(x)$, read as "function of x". The set of all the values of the function is called the ***range*** of the function.

A symbol such as x or y, used to represent an arbitrary element of a set is called a ***variable***. The symbol x which represents an element in the domain is called the ***independent variable***, and the symbol y which represents the element corresponding to x is called the ***dependent variable*** since the value of y depends upon the chosen value of x.

We give below a few examples to illustrate the concept of function.

Ex. 1. The function f defined by $X = \{x \mid 1 < x < 5\}, Y = \{y \mid -2 < y < 6\}$ and the rule $y = 8 - 2x$, is

$$f = \{(x, y) \mid x \in X, y = 8 - 2x\}. \qquad (1)$$

The domain of the function is the interval $(1, 5)$, and the range of the function is $(-2, 6)$.

The function f defined by $X = \{x \mid 1 < x < 5\}, Y = \{y \mid -\infty < y < \infty\}$ and the rule $y = 8 - 2x$ is again given by (1). In this case also one element of Y corresponds to each element of the domain $(1, 5)$. Of course in this case there are in Y a number of extra elements which do not correspond to any element in the domain. The range, which is the set of all elements in Y corresponding to the elements of X, is again $(-2, 6)$.

We therefore see that it is not necessary to specify the set Y. The set X and the rule $y = 8 - 2x$ are sufficient to determine the function f.

Ex. 2. The rule :

$$y = x - 1 \text{ when } x \leq 2, \text{ and } y = x^2 \text{ when } x > 2$$

defines a single function for the domain consisting of all real numbers, because to each x corresponds a single y. It does not matter that y is calculated from different formulae in different parts of the domain.

Ex. 3. The rule $y^2 = x - 1$ gives two value of y for each $x > 1$, viz.

$$y = +\sqrt{x-1} \text{ and } y = -\sqrt{x-1}$$

These respectively define the two functions

$$f_1 = \{(x, y) | \; y = +\sqrt{x-1}, \; x \geq 1\}$$

and $$f_2 = \{(x, y) | \; y = -\sqrt{x-1}, \; x \geq 1\}$$

The union of these two sets, viz.

$$\{(x, y) | \; y^2 = x - 1, \; x \geq 1\}$$

is not a function but a relation, since it gives more than one value of y corresponding to any x.

Such a relation is sometimes called a ***double-valued*** (or ***multiple-valued***) function.

Ex. 4. The sets

$X = \{x \,|\, x$ is the first name of any student in a given class$\}$,

and $Y = \{y \,|\, y$ is the surname of the student $x\}$,

define a function, provided no two students in that class have the same first name. The conditions for a function are satisfied here since to each first name x corresponds a single surname y.

Of course, if the class contains two students Ram Prasad and Ram Dulare with the same first name 'Ram', then the conditions for a function are not satisfied as two elements of Y (Prasad and Dulare) correspond to the same element of X (Ram).

This example of a function may appear strange to the student, but it brings out the fact that the definition of a function given above is very general. This is useful in as much as the same definition serves the purpose in different branches of analysis.

1.61. Remarks. (1) It will be clear from the above examples that two things are necessary to completely determine a function: (i) its ***domain***, which states exactly which elements x are to be used as first members of the ordered pairs constituting the function, and (ii) the ***rule*** which determines the corresponding element y that is to be paired with each x. The range of the function can always be found from the given domain and rule.

(2) In the definition of a function, the value of y corresponding to each x should be perfectly determinate. Values like 0/0, 5/0, a/0 have no meaning. Some students are under the impression that 5/0 is "infinity". This is wrong. Infinity is not a number, and has no meaning except in the sense defined later in § 2.5. Nor is 0/0 equal to 1. If we suppose $0/0 = 1$, we shall be led to absurdities. Consider, for example, what would be the result of dividing by zero both sides of an equation like $5 \times 0 = 2 \times 0$.

Again an expression like $$\frac{x^2 - 1}{x - 1}$$

fails to give a definite value when $x = 1$. The numerator and the denominator both become zero when $x = 1$. The student must not think that we can get the value of $(x^2 - 1)/(x - 1)$ by first dividing the numerator by the denominator and then putting $x = 1$; for *we can divide by* $x - 1$ *only when* x *is not equal to* 1.

For this reason a rule like $y = 1/x$ does not define a function completely in any domain which includes 0, since it does not state the value of y corresponding to $x = 0$. In such a case we say that ***the function is not defined for*** $x = 0$.

1.7. Some special functions. We shall only define some special functions assuming that the student is familiar with the commonly used functions such as the exponential, logarithmic, trigonometric and inverse functions.

(1) **Greatest integer function.** The function $f(x) = [x]$ is defined as the greatest integer if it is either less than or equal to x. For example,

$$[3] = 3, \; [0.843] = 0, \; [-1.4] = -2, \; [-5] = -5.$$

Thus, we see that if n be any integer, then

$$[x] = n \quad \text{where} \quad n \le x < n+1.$$

The domain of this function is the set of real numbers R and its range is the set of integers I.

(2) **Least integer function.** This function $f(x) = (x)$ is defined as the least integer if it is either equal to or greater than x. For example,

$$(3) = 3,\ (0.843) = 1,\ (-1.4) = -1,\ (-5) = -5.$$

Thus, we see that if n be any integer, then

$$(x) = n+1 \qquad \text{where} \qquad n < x \le n+1.$$

The domain of this function is the set of real numbers R and its range is the set of integers I.

(3) **Signum function.** We define this function [written as sgn (x)] as

$$sgn\,(x) = \begin{cases} x/|x|, & x \ne 0 \\ 0, & x = 0 \end{cases} \quad \text{or} \quad sgn\,(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

using the definition of $|x|$. The domain of this function is the set of real numbers R and its range is the set $[-1, 0, 1]$.

(4) **Composite function (or function of a function).** If f and g be two functions such that *the range of f is a subset of the domain of g*, then the composite function $g \circ f$ (taken in that order) is defined by

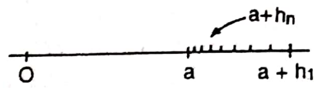
$$(g \circ f)(x) = g[f(x)] \text{ for all } x \in \text{domain of } f.$$

The student should note that, generally, $g \circ f \ne f \circ g$.

2

Limits and Continuity

2.1. Limit of a function. Let f be a function given by the rule $y = f(x)$. Choose any set of positive numbers $h_1, h_2, h_3, \ldots, h_n, \ldots,$ which continually decrease, i.e.,



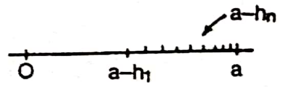
$$h_1 > h_2 > h_3 > \ldots > h_n > \ldots > 0$$

and can be made as small as we please by taking n large enough. It may then happen that the values

$$f(a+h_1),\ f(a+h_2),\ f(a+h_3), \ldots,\ f(a+h_n), \ldots$$

of the function continually approach a number A as h_n gets smaller and smaller. This number A is called the *"limit of $f(x)$ at a"*.

In fact A is the ***limit on the right***, since we have considered only the values of x greater than a (i.e. on the right of a in the figure). If we consider the values of the function:



$$f(a-h_1),\ f(a-h_2),\ f(a-h_3), \ldots,\ f(a-h_n), \ldots$$

and find that they continually approach a number B as h_n gets smaller and smaller, we call B the ***limit of $f(x)$ on the left***.

When $A = B$ (as happens in many cases), we call A the limit of $f(x)$ at a.

For the limit to exist, it is important that $f(a+h_n) - A$ can be made numerically as small as we like by taking h_n small enough.

Putting $a+h_n$ or $(a-h_n)=x$, and noting that $|x-a|=h_n$ we can give the following definition of the limit of a function.

If $|f(x)-A|$ can be made as small as we please for all values of x for which $|x-a|$ is sufficiently small (but not zero), we say that $f(x)$ has the limit A as $x \to a$.

We write $\lim_{x\to a} f(x)=A$

which is read as " *the limit of $f(x)$ as x tends to a is A* ".

2.2. $\varepsilon-\delta$ definition of a limit. Let ε be any arbitrarily chosen positive number, however small. Then for $f(x)$ to approach its limit A, $|f(x)-A|$ must be less than ε. The number ε being arbitrary can be taken as small as we please to suit our requirements.

The question remains: for what values of x should $|f(x)-A|<\varepsilon$ hold. We see that as $|x-a|$ decreases, so does $|f(x)-A|$.

For a $\delta>0$, there will be a neighbourhood of a, from $a-\delta$ on one side of a to $a+\delta$ on the other side, such that $|f(x)-A|<\varepsilon$ whenever x lies between $a-\delta$ and $a+\delta$ which can be concisely stated as $|x-a|<\delta$.

The value of δ will depend on ε, and of course on the function f. The value of δ *should not be zero*, because if $\delta=0$, the neighbourhood vanishes, and no value of $f(x)$ remains to approach A if $f(x)$ is not defined at $x=a$.

Hence the $\varepsilon-\delta$ definition of a limit is as follows:

If f is a function of the variable x, the number A is said to be the limit of $f(x)$ as $x\to a$ if, for any arbitrarily chosen positive number ε, however small but not zero, there exists a corresponding number $\delta>0$ such that **$|f(x)-A|<\varepsilon$, for all values of x for which**

$$0<|x-a|<\delta.$$

From the definition it is clear that the limit of $f(x)$ as $x\to a$ has no necessary connection with the value of the function at a, except that when the function is continuous at $x=a$, the two are equal; and even in the case of such functions the limit is distinct in idea from the value.

2.21. Left hand and right hand limits.

Left hand limit. A number A is said to be the ***left hand limit*** of $f(x)$ as x tends to a *from the left* if, for any arbitrarily chosen $\varepsilon>0$, however small, there exists a corresponding number $\delta>0$ such that

$|f(x)-A|<\varepsilon$, for all values of x for which $a-\delta<x<a$.

We write this as $\lim_{x\to a-0} f(x)=A$ or $f(a-0)=A$.

Right hand limit. A number A is said to be the ***right hand limit*** of $f(x)$ as x tends to a *from the right* if, for any arbitrarily chosen $\varepsilon>0$, however small, there exists a corresponding number $\delta>0$ such that

$|f(x)-A|<\varepsilon$, for all values of x for which $a<x<a+\delta$.

We write this as $\lim_{x\to a+0} f(x)=A$ or $f(a+0)=A$.

It may be noted that the ***limit of $f(x)$ exists only if both the left hand and right hand limits exist and are equal to each other*** otherwise not.

2.22. Some theorems about limits. The truth of the following theorems may be taken for granted.

If f and g be two functions such that $\lim_{x\to a} f(x)=A$ and $\lim_{x\to a} g(x)=B$ where A and B are finite, then,

(i) $\lim_{x\to a}(k\,f)x=k\,A$ where k is a constant.

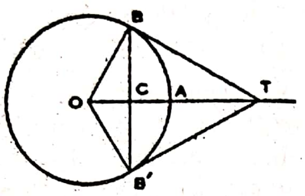
(ii) $\lim_{x\to a}(f\pm g)x=A\pm B$. (iii) $\lim_{x\to a}(f\,g)x=A.B$

(iv) $\lim_{x\to a}(f/g)x=A/B$ provided that $B\neq 0$.

These theorems are helpful in finding limits. They can be easily extended to cover cases where more than two functions are involved.

2.3. An important limit. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

To prove this, consider a circle BAB' with centre O such that $\angle AOB = x$ where $0 < x < \frac{1}{2}\pi$. Let the tangents to the circle at B and B' meet OA in T and the chord BB' intersect OA in C. Then, from the figure, we see that



$$\text{Chord } BC < \text{Arc } BA < BT \qquad (1)$$

Now, in right-angled triangle OBC, chord $BC = OB\sin x$.

In right-angled triangle OBT, $BT = OB\tan x$ and in sector OAB, arc $BA = OB.x$. Substitution in (1) gives

$$\sin x < x < \tan x \quad \text{or} \quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad (2)$$

on dividing by $\sin x$ which is positive since $0 < x < \frac{1}{2}\pi$.

Taking reciprocals of (2), $1 > \dfrac{\sin x}{x} > \cos x$ (3)

We have obtained this result for $x > 0$ but the above inequality holds true even for $x < 0$ when $-\frac{1}{2}\pi < x < 0$ since

$$\sin(-x)/(-x) = \sin x/x \text{ and } \cos(-x) = x.$$

Now $\lim\limits_{x\to 0}\cos x = 1$, so taking limits as $x \to 0$, (3) gives

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Corollary. An important corollary which follows from the above result is $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = 1$.

Now $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}.\dfrac{1}{\cos x}\right) = 1\times 1 = 1$.

Ex. 1. Examine whether the limit of the function $\dfrac{x^2-1}{x-1}$ as x tends to 1 exists. [*Avadh* 2009]

Let $f(x) = \lim_{x\to 1}\dfrac{x^2-1}{x-1}$.

The function is not defined for $x = 1$. If $x \neq 1$, then $\dfrac{x^2-1}{x-1} = x+1$ So we see that as x tends to 1, $f(x)$ approaches 2. Now when $x \neq 1$,

$$|f(x)-2| = |(x+1)-2| = |x-1|.$$

Thus for any arbitrarily chosen number $\varepsilon > 0$,

$$|f(x)-2| < \varepsilon \qquad \text{whenever} \qquad |x-1| < \varepsilon. \qquad (1)$$

If we choose $\delta > 0$, such that $\delta = \varepsilon$, then by (1), we have

$$|f(x)-2| < \varepsilon \qquad \text{whenever} \qquad |x-1| < \delta.$$

Hence $\lim_{x\to 1}\dfrac{x^2-1}{x-1} = 2$.

Ex. 2. Find $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ [*Lucknow* 2012]

Rationalising the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\cdot\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \lim_{x\to 0}\frac{[(1+x)-(1-x)]}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{1+1} = 1.$$

Ex. 3. Find $\lim\limits_{x\to\infty}\dfrac{x^2+3x-2}{2x^2+3}$.

$$\lim_{x\to\infty}\frac{x^2+3x-2}{2x^2+3} = \lim_{x\to\infty}\frac{1+(3/x)-(2/x^2)}{2+(3/x^2)} = \frac{1}{2},$$

since $3/x$, $2/x^2$ and $3/x^2$ tend to zero as $x \to \infty$.

Ex. 4. Show that $\lim_{x\to 0}\dfrac{e^x-1}{x} = 1$

Using the expansion of the exponential series e^x, we get

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{\left[1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+....\right]-1}{x}$$

$$= \lim_{x\to 0}\left[1+\frac{x}{2!}+\frac{x^2}{3!}+\dots\right]=1. \qquad \text{(dividing throughout by } x\text{)}$$

Ex. 5. Find $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}$.

Putting $x=\tan\theta$, we see that as $x\to 0$, we have $\theta\to 0$. Hence

$$\lim_{x\to 0}\frac{\tan^{-1}x}{x}=\lim_{\theta\to 0}\frac{\tan^{-1}(\tan\theta)}{\tan\theta}=\lim_{\theta\to 0}\frac{\theta}{\tan\theta}=\frac{1}{\lim\limits_{\theta\to 0}(\tan\theta/\theta)}=1.$$

Ex. 6. Evaluate $\lim\limits_{x\to 0} x/|x|$.

Using the definition of $|x|$, we have

$$f(x)=\lim_{x\to 0}|x|/x=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0.\end{cases} \qquad (1)$$

Let h be a positive number, then

$$f(0-0)=\lim_{h\to 0}f(0-h)=-1 \qquad \text{[from (1)]}$$

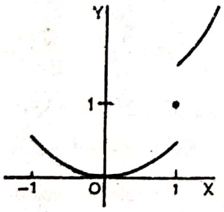
and $$f(0+0)=\lim_{h\to 0}f(0+h)=1. \qquad \text{[from (1)]}$$

Since $f(0-0)\neq f(0+0)$, the limit of $f(x)$ does not exist.

NOTE. The student should note that in finding the limit at a we do not use $f(a)$, the value of $f(x)$ at a. The limit depends on the values of $f(x)$ at points in the *neighbourhood* of $x=a$, other than a itself.

2.4. A limit does not necessarily exist. An example is the function defined as follows :

$y=\frac{1}{2}x^2$ when $x<1$,
$y=1$ when $x<1$,
$y=1+\frac{1}{2}x^2$ when $x>1$,



The graph of the function consists of two different parabolas and an isolated point (1, 1). The student can easily verify that the limit from the left, $f(1-0)=\frac{1}{2}$ and the limit from the right, $f(1+0)=\frac{3}{2}$. Since the limits on the left and right are unequal, $\lim\limits_{x\to 1} y$ does not exist.

There are also functions for which for certain values of x, neither the limit on the right nor the limit on the left exists. This is illustrated by the following example.

Ex. Prove or disprove if the function defined by $f(x)=\sin(1/x)$ has a limit at $x=0$. **[*Allahabad* 2013]**

The function is defined for all values of x except $x=0$. To draw the graph, we first consider the positive values of x starting from ∞ and tending to zero.

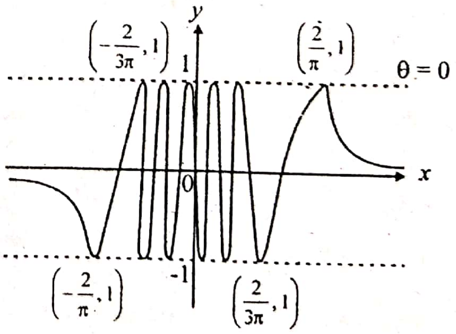
We see that as x decreases from ∞ to $2/\pi$, $1/x$ increases from 0 to $\pi/2$ i.e. $\sin(1/x)$ increases from 0 to 1.

In the interval $2/\pi$ to $1/\pi$, $\sin(1/x)$ varies from 1 to 0. Again, in the interval $1/\pi$ to $2/3\pi$, $\sin(1/x)$ varies from 0 to -1, in $2/3\pi$ to $1/2\pi$, it varies from -1 to 0 and so on.

It follows that as x tends to zero from the right, the value of $\sin(1/x)$ oscillates between + 1 and -1, passing through 0 and the intermediate values an indefinitely large number of times. We can never hope to draw the complete graph up to $x=0$, but a portion is shown in the figure.

Evidently there is no number A to which $\sin(1/x)$ tends to as $x\to 0$ from the right. So the limit on the right does not exist. Similarly, the limit on the left also does not exist.

Hence the ordinary limit of $\sin(1/x)$, without any qualification as to right or left, also does not exist as $x\to 0$.



2.5. Infinite Limits. Consider the function defined by $y = 1/x^2$ for every value of x.

When x is very small, y is very large. If we take smaller and smaller values of x, y gets larger and larger. As x approaches 0 (either from the right or the left) there is no barrier to the increase of $1/x^2$. We cannot specify a number N and say that $1/x^2$ will remain less than N; for, however large N may be we can make $1/x^2$ greater than N by choosing x small enough.

We say, therefore, that $1/x^2$ *tends to infinity* (*or the limit of* $1/x^2$ *is infinity*) *as* x *tends to zero.*

Thus, ***a function f is said to tend to infinity as*** $x \to a$ ***if for any positive number N, however large, there exists a*** $\delta > 0$ ***such that***

$$f(x) > N \qquad \textit{for all} \qquad x \in (a-\delta, a+\delta).$$

NOTE. The expressions "x tends to infinity" does not mean that infinity is some number like 1 million or 1 billion, only very much larger. The expression "*x tends to infinity" means merely that x goes on increasing, and there exists no number N compared to which all these values of x are less.*

" $1/x^2 = \infty$ when $x = 0$ " is an erroneous way of saying that $\lim_{x \to 0} (1/x^2) = \infty$.

EXAMPLES

1. By using the $\varepsilon - \delta$ method, prove that

(i) $\lim_{x \to 3} (x^2 + 2x) = 15.$ [*Bundelkhand* 06, *Bilaspur* 08]

(ii) $\lim_{x \to 0} x^2 \sin (1/x) = 0.$ [*Bilaspur* 2010]

Evaluate the following limits

2. $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$.

3. $\lim_{x \to 0} \dfrac{\sin ax}{\sin bx}$.

4. $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$. [*Lucknow* 2007]

5. $\lim_{x \to 0} \dfrac{|x|}{x}$. [*Lucknow* 2011]

6. $\lim_{x \to 0} \dfrac{3 \sin^{-1} x}{4x}$.

7. $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$.

[Hint: Use $\dfrac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2 x^{n-3} + \ldots + a^{n-1}$]

8. Prove or disprove that if $f(x) = x \sin (1/x)$, $x \neq 0$, then $\lim_{x \to 0} f(x) = 0$. [*Avadh* 08, *Allahabad* 10]

9. Discuss the existence of $\lim_{x \to 0} x \tan^{-1} (1/x)$.

10. If $f(x) = \begin{cases} x, & 0 \le x < \frac{1}{2} \\ 0, & x = 1/2 \\ x - 1, & \frac{1}{2} < x \le 0. \end{cases}$ Does $\lim_{x \to 1/2} f(x)$ exist?

ANSWERS

2. 0. **3.** a/b. **4.** $1/2$. **5.** Limit does not exist

6. $3/4$. **7.** $n a^{n-1}$. **8.** Limit is 0.

10. Does not exist since $f(\frac{1}{2} - 0) = \frac{1}{2}$ and $f(\frac{1}{2} + 0) = -\frac{1}{2}$.

2.6. Continuity. If we draw the graph of the function defined by $y = x^2$, we find that the graph is continuous, i.e. it has no breaks in it. But the graph of the function defined as in § 2.4, has a discontinuity at the point $x = 1$ (see first fig.).

So, we see that a discontinuity implies a sudden jump in the value of a function, whereas the value of a continuous function alters only gradually for gradual alterations in x.

So for a curve to be continuous at $x = a$, it should be possible to make the numerical difference $|f(x) - f(a)|$ arbitrarily small by choosing x near enough to a.

This gives us the following precise definition.

A function f of a real variable x is said to be CONTINUOUS at $x = a$ *if for any arbitrarily chosen* $\varepsilon > 0$, ***however small (but not zero), we can find a corresponding*** $\delta > 0$ ***such that***

$$| f(x) - f(a) | < \varepsilon \quad \text{for all values of } x \text{ for which}$$
$$0 < | x - a | < \delta.$$

Comparing this definition with that of the limit, we see that ***for a function to be continuous at*** $x = a$,

$$\lim_{x \to a} f(x) = f(a).$$

When $\lim f(x) \neq f(a)$, or when the limit does not exist or is infinite, the function is ***discontinuous*** at $x = a$.

Continuity from the left. A function f is said to be continuous from the left at $x = a$ if $\lim\limits_{x \to a-0} f(x)$ exists and is equal to $f(a)$.

Continuity from the right. A function f is said to be continuous from the right at $x = a$ if $\lim\limits_{x \to a+0} f(x)$ exists and is equal to $f(a)$.

Thus a function f is continuous at $x = a$ if $\lim\limits_{x \to a-0} f(x)$ and $\lim\limits_{x \to a+0} f(x)$ both exist and are finite, and

$$\lim_{x \to a-0} f(x) = f(a) = \lim_{x \to a+0} f(x)$$

or $$f(a-0) = f(a) = f(a+0).$$

Otherwise the function is discontinuous at $x = a$.

Ex. 1. $\sin(1/x)$ is discontinuous at $x = 0$, since $\lim\limits_{x \to 0} \sin(1/x)$ does not exist.

Ex. 2. $1/x^2$ is discontinuous at $x = 0$, since $\lim\limits_{x \to 0} (1/x^2)$ is not finite.

2.61. Geometrical Meaning of Continuity. As stated earlier, the graph of a continuous function will have no breaks in it.

When f is continuous at $x = a$, we can find an appropriate $\delta > 0$ for any given $\varepsilon > 0$ and see graphically that for all x lying in the δ - neighbourhood of a, i.e. $(a-\delta, a+\delta)$, the curve of the function will lie in the ε - neighbourhood of $f(a)$, i.e. $(f(a)-\varepsilon, f(a)+\varepsilon)$.

The dependence of δ on ε is illustrated in fig.1 (on the next page). We see that when f is continuous we can find appropriate δ for any given ε.

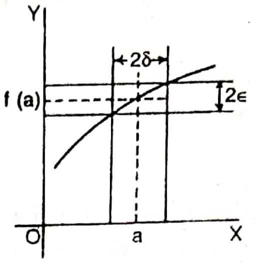


Fig. 1

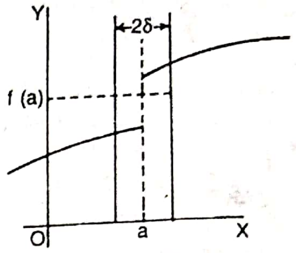


Fig. 2

This is not so when f is discontinuous at a (fig. 2). At such a point, however small we choose our δ, we cannot make $|f(x) - f(a)|$ arbitrarily small for all x lying between $a - \delta$ and $a + \delta$.

2.62. Some theorems about continuity. If two functions are continuous at a point, then their sum, difference, product and quotient will also be continuous at that point.

If f and g be two functions which are continuous at $x = a$, then the following functions will also be continuous at $x = a$:

(i) $f \pm g$ (ii) $f\,g$ (iii) $k\,f$ (k = constant)

(iv) $f \,/\, g$ where $\lim\limits_{x \to a} g(x) = g(a) \neq 0$.

The proofs of these are left to the student.

2.63. Continuous Functions. *A function f is said to be* ***continuous in the interval (a, b)*** *if it is continuous for every value of x in the interval (a, b).*

A function f is said to be ***bounded*** *in the interval (a, b) if there exist numbers A, B such that*

$$A \leq f(x) \leq B$$

for every value of x in the interval (a, b).

In the above definitions we can also replace (a, b) by the closed interval $[a, b]$.

A function f defined on a finite interval is **bounded** unless the limit of $f(x)$ at any point is infinite. Therefore a function continuous in the closed interval $[a, b]$ will be bounded. This may not hold for a function continuous in an open interval; e.g. $f(x)=1/x$ is continuous in the interval (0, 1). but is not bounded since $\lim_{x \to 0+} f(x)=\infty$.

It can be easily verified (or seen from their graphs) that

(i) x^n is continuous for all values of x when n is positive.

(ii) $1/x^n$ (n positive) is continuous for all values of x except $x=0$.

(iii) e^x, $\sin x$ and $\cos x$ are continuous for all values of x; $\log x$ is continuous for all values of $x>0$. (For $x<0$, $\log x$ is not real.)

(iv) $\tan x$ is continuous for all values of x except $\pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$, $\pm\frac{5}{2}\pi$, etc. As $x \to \frac{1}{2}\pi$ from the right, $\tan x \to -\infty$. As $x \to \frac{1}{2}\pi$ from the left, $\tan x \to +\infty$. Thus $\lim_{x \to \pi/2} \tan x$ does not exist.

Similar statements can be made for $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

(v) The inverse circular functions have an indefinitely large number of values for every value of x. For the sake of convenience, throughout this book, we shall suppose that $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$ and $\operatorname{cosec}^{-1} x$ lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ (both values inclusive); and $\cos^{-1} x$, $\sec^{-1} x$ lie between 0 and π (both values inclusive).

$\sin^{-1} x$ and $\cos^{-1} x$ (sometimes written as arc sin x and arc cos x respectively) are not real for values of x outside the domain [-1,1]. Similarly $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ are not real for values of x between -1 and 1 (both values exclusive).

All the inverse circular functions are continuous at every point within the domains in which they are real.

Ex. 1. Show that $\sin x$ is continuous for every value of x.

Let a be any value of x; then

$$|\sin x-\sin a|=|2\sin \tfrac{1}{2}(x-a)\cos \tfrac{1}{2}(x+a)|<2|\sin \tfrac{1}{2}(x-a)| \qquad (1)$$

because $|\cos \frac{1}{2}(x-a)|<1$.

Now when θ lies between 0 and $\frac{1}{2}\pi$ we have $\sin\theta \le \theta$. Therefore, restricting x to values for which $|\frac{1}{2}(x-a)| < \frac{1}{2}\pi$, we see from (1) that

$$|\sin x-\sin a|<2|\tfrac{1}{2}(x-a)|.$$

Therefore $\quad |\sin x-\sin a|<\varepsilon \quad$ if $\quad |x-a|<\varepsilon.$

[We have thus found δ; it is equal in this case to ε itself.]

This is true however small ε may be. Hence $\sin x$ is continuous at $x=a$, whatever a may be.

Hence $\sin x$ is continuous for every value of x.

Ex. 2. Show that x^n, where n is a positive integer, is continuous at $x=a$.

The value of x^n at $x=a$ is a^n; then

$$\begin{aligned}|x^n-a^n|&=|x-a|.|x^{n-1}+x^{n-2}a+\ldots+a^{n-1}|\\&\le|x-a|.(|x^{n-1}|+|x^{n-2}a|+\ldots+|a^{n-1}|)\\&<|x-a|.nb^{n-1} \qquad (1)\end{aligned}$$

where b is a fixed positive number greater than both $|x|$ and $|a|$.

Take an arbitrary positive number ε, and choose $\delta=\varepsilon/nb^{n-1}$. Then

by (1), $\quad |x^n-a^n|<\varepsilon \quad$ whenever $\quad |x-a|<\delta$

Hence x^n is continuous at $x=a$.

Ex. 3. Examine the continuity the function

$$f(x)=\begin{cases}-x^2, & x\le 0\\ 5x-4, & 0<x\le 1\\ 4x^2-3x, & 1<x<2\\ 3x+4, & x\ge 2\end{cases} \quad \text{at } x=0, 1, 2.$$

[***Meerut*** **04,** ***Purvanchal*** **04,** ***Allahabad*** **05,** ***Lucknow*** **06,** ***Avadh*** **06,** ***Delhi*** **07]**

Let h be a positive number, then

at $x=0$, $\quad f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}[5(0+h)-4]=-4$

and $\quad f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}[-(0-h)^2]=0.$

Since $\quad f(0+0)\ne f(0-0)$, f is not continuous at $x=0$.

At $x=1$,

$f(1+0)=\lim_{h\to 0} f(1+0)=\lim_{h\to 0}[4\,(1+h)^2-3\,(1+h)]=1$

and $f(1-0)=\lim_{h\to 0} f(1-0)=\lim_{h\to 0}[5\,(1-h)-4]=1$.

Also, $f(1)=[5x-4]_{x=1}=1$.

Since $f(1+0)=f(1-0)=f(1)$, f is continuous at $x=1$.

At $x=2$, $f(2+0)=\lim_{h\to 0} f(2+0)=\lim_{h\to 0}[3\,(2+h)+4]=10$.

and $f(2-0)=\lim_{h\to 0} f(2-0)=\lim_{h\to 0}[4\,(2-h)^2-3\,(2-h)]=10$

Also, $f(2)=[3x+4]_{x=2}=10$.

Since $f(2+0)=f(2-0)=f(2)$, f is continuous at $x=2$

Ex. 4. Discuss the continuity of the following function at $x=a$

$$f(x)=\begin{cases}(x^2/a)-a, & \text{for } 0<x<a\\ 0, & \text{for } x=0\\ a-(a^3/x^2), & \text{for } x>0.\end{cases}$$

[*Avadh* **07, 09,** *Agra* **09,** *Kashi Vidyapeeth* **08,** *Lucknow* **12**]

Let h be a positive number, then at $x=a$, we have

$$f(a+0)=\lim_{h\to 0} f(a+h)=\lim_{h\to 0}\left\{a-\frac{a^3}{(a+h)^2}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{a\,(a^2+2ah+h^2)-a^3}{(a+h)^2}\right\}=\lim_{h\to 0}\frac{2a^2h+ah^2}{(a+h)^2}=0$$

and

$$f(a-0)=\lim_{h\to 0} f(a-h)=\lim_{h\to 0}\left\{\frac{(a-h)^2}{a}-a\right\}$$

$$=\lim_{h\to 0}\frac{h^2-2ah}{a}=0.$$

Since $f(a+0)=f(a-0)=f(a)$, f is continuous at $x=a$.

EXAMPLES

1. Show that the function $f(x)=3x^2+2x-1$ is continuous for $x=2$. [*Lucknow* **2008**]

2. Show that the function $f(x)=|x|$ is continuous at $x=0$. [*Rohilkhand* **07,** *Avadh* **11**]

3. Is the following function continuous at $x=2$?

$$f(x)=\begin{cases}1+x, & \text{when } x\le 2\\ 5-x, & \text{when } x>2.\end{cases}$$

[*Meerut* **05,** *Lucknow* **09,** *Bundelkhand* **09**]

4. Discuss the continuity of the function defined by

$f(x)=|x|+|x-1|$ at $x=0$ and 1.

[*Delhi* **08,** *Kashi Vidyapeeth* **10**]

Are the following functions continuous at $x=0$?

5. $f(x)=\begin{cases}\cos(1/x), & \text{when } x\neq 0.\\ 0, & \text{when } x=0.\end{cases}$ [*Lucknow* **05,** *Allahabad* **09**]

6. $f(x)=\begin{cases}x\sin(1/x), & \text{when } x\neq 0.\\ 0, & \text{when } x=0.\end{cases}$

7. Prove that the function $f(x)=\dfrac{1}{x}\sin^{-1}x$ for $x\neq 0$, $f(0)=1$ is continuous at the origin. [*Agra* **08, 10**]

8. Show that the function $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{for } x\neq 0.\\ 1, & \text{for } x=0.\end{cases}$ is continuous at the origin. [*Avadh* **2008**]

9. Examine the continuity of the function $f(x)=\dfrac{1}{1-e^{1/x}}$ at $x=0$. [*Lucknow* **2010**]

10. Show that the following function is discontinuous at $x=\frac{1}{2}$,

$$f(x)==\begin{cases}x & \text{when } 0\le x<\frac{1}{2}\\ 1 & \text{when } x=\frac{1}{2}\\ 1-x & \text{when } \frac{1}{2}<x<1.\end{cases}$$

ANSWERS

3. Continuous.
4. Continuous at $x=0$ and 1.
5. Discontinuous.
6. Continuous.
9. Discontinuous.

2.7. Types of discontinuity. We have already mentioned discontinuity at a point. We shall now list the different types of discontinuities.

(1) **Removable discontinuity.** A function f has a removable discontinuity at a point a if $\lim_{x \to a} f(x)$ exists but

$$\lim_{x \to a} f(x) \neq f(a) \quad \text{i.e.} \quad f(a-0) = f(a+0) \neq f(a).$$

This discontinuity can be removed by changing the value of $f(a)$ such that $\lim_{x \to a} f(x) = f(a)$.

(2) **Discontinuity of the first kind.** A function f is said to have a discontinuity of the first kind at a point a if limits from both the left and right exist but are not equal.

i.e. $$f(a-0) \neq f(a+0).$$

f has a *discontinuity of the first kind* at a point a

from the left if $f(a-0) \neq f(a)$ but $f(a+0) = f(a)$ and

from the right if $f(a+0) \neq f(a)$ but $f(a-0) = f(a)$.

(3) **Discontinuity of the second kind.** A function f is said to have a discontinuity of the second kind at a point a if neither of the limits $f(a-0)$ or $f(a+0)$ exist.

f has a *discontinuity of the second kind* at a point a

from the left if $f(a-0)$ does not exist and

from the right if $f(a+0)$ does not exist.

Ex. 1. Examine the continuity of the following function at the origin

$f(x) = \dfrac{\sin^2 ax}{x^2}$ for $x \neq 0$, $f(0) = 1$. [*Meerut* **10**, *Lucknow* **12**]

We can write f as $$f(x) = \frac{\sin^2 ax}{x^2} = a^2 \left[\frac{\sin ax}{ax}\right]^2$$

For a positive number $h > 0$, we have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} a^2 \left[\frac{\sin a(0+h)}{a(0+h)}\right]^2$$

$$= a^2 \lim_{ah \to 0} \left[\frac{\sin ah}{ah}\right]^2 = a^2 \quad \text{since } ah \to 0 \text{ as } h \to 0$$

and $\lim_{\theta \to 0} (\sin\theta/\theta) = 1.$

Similarly $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} a^2 \left[\frac{\sin a(0-h)}{a(0-h)}\right]^2$$

$$= a^2 \lim_{ah \to 0} \left[\frac{-\sin ah}{-ah}\right]^2 = a^2.$$

Since $f(0+0) = f(0-0)$, hence $\lim_{x \to 0} f(x) = a^2$ but $f(0) = 1$. So, the function has a removable discontinuity at $x = 0$ which can be removed by taking $f(0) = a^2$.

Ex. 2. Discuss the continuity of $f(x) = [x]$ at $x = 0$.

[*Rohilkhand* **2008**]

Using the definition of the greatest integer function, we have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} [0+h] = 0$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} [0-h] = -1.$$

Also $$f(0) = [0] = 0.$$

Since $f(0+0) = f(0) \neq f(0-0)$, hence the function has a discontinuity of the first kind from the left at $x = 0$.

Ex. 3. Discuss the continuity at $x = 0$ of the following function

$$f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1} \quad \text{when} \quad x \neq 0$$

$$= 1 \quad \text{when} \quad x = 0.$$

[*Bilaspur* **08, 09**, *Kumaon* **10**, *Kanpur* **11**]

Let h be a positive number, then

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{1/(0+h)} - 1}{e^{1/(0+h)} + 1}$$

$$= \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h \to 0} \frac{1 - e^{-1/h}}{1 + e^{-1/h}} = 1 \quad \text{on}$$

dividing both the numerator and denominator by $e^{1/h}$. Also as $h \to 0$, both $1/h$ and $e^{1/h}$ tend to ∞. So $e^{-1/h} \to 0$.

Again $f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\dfrac{e^{1/(0-h)}-1}{e^{1/(0-h)}+1}$

$$=\lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1}=-1 \text{ as before .}$$

Since $f(0+0)=f(0)\neq f(0-0)$, hence the function has a discontinuity of the first kind from the left at $x=0$.

EXAMPLE

1. Test the continuity of a function at $x=0$ which is defined as follows: $f(x)=\begin{cases}\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0.\end{cases}$

2. Examine the continuity of the function $f(x)=\dfrac{\tan 2x}{x}, x\neq 0,$ $f(0)=1$ at $x=0.$

3. Test the continuity of the function

$$f(x)=\frac{1}{x-a}\operatorname{cosec}\left(\frac{1}{x-a}\right),\quad x\neq a,\quad f(a)=0.$$

[*Lucknow* 08, *Bundelkhand* 08]

What kind of discontinuity does the function have?

[*Kashi Vidyapeeth* 2011]

4. Test the continuity of the following function at the origin

$$f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 1, & x=0.\end{cases}$$ [*Kanpur* 08, *Raipur* 15]

5. Examine the continuity of the function $f(x)=\dfrac{\sqrt{1-\cos 2x}}{x}$, $x\neq 0$, $f(0)=0$ at $x=0$. In case of discontinuity, state the kind of discontinuity. [*Himachal* 2005]

6. Show that the function

$$f(x)=\begin{cases}\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & x\neq 0.\\ 0, & x=0.\end{cases}$$ [*Delhi* 2006]

is discontinuous at $x=0$. Also mention the type of discontinuity.

7. Show that the function $f(x)$ defined by

$$f(x)==\begin{cases}0, & \text{for} & x=0\\ \frac{1}{2}-x, & \text{for} & 0<x<\frac{1}{2}\\ \frac{1}{2}, & \text{for} & x=\frac{1}{2}\\ \frac{3}{2}-x, & \text{for} & \frac{1}{2}<x<1\\ 1, & \text{for} & x=1.\end{cases}$$

has three points of discontinuity. Find such points and examine the kind of discontinuities. [*Meerut* 05, *Allahabad* 06, *Indore* 07]

ANSWERS

1. Discontinuity of the second kind at $x=0$.
2. Removable discontinuity.
3. Discontinuity of the second kind at $x=a$.
4. Discontinuity of the first kind from the left at $x=0$.
5. Removable discontinuity.
6. Discontinuity of the first kind.
7. Discontinuity of the first kind at $x=1/2$; removable discontinuity at $x=0, 1$.

3

Differentiation

3.1. Definition. ***Let f be a function defined for all real values of x in an interval L. For $a \in L$, if***

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \textit{or} \quad \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists, then this limit is called the differential coefficient of $f(x)$ at the point $x = a$. It is generally denoted by $f'(a)$.

The differential coefficient is also called the ***derivative*** or the ***derived function***. The process of finding the differential coefficient is called ***differentiation***. In simple terms, the derivative of a function measures the rate of change of the function.

We have seen that a limit may exist or may not. The differential coefficient, therefore, may exist at $x = a$ or may not. If it exists, we say that the function is ***differentiable*** at $x = a$.

Notation. The differential coefficient of $f(x)$ is written generally as $\frac{d}{dx}f(x)$ or $f'(x)$ or $Df(x)$.

The differential coefficient at the point $x = a$ is generally represented by $f'(a)$ or $\{df(x)/dx\}_{x=a}$.

3.11. Left hand and right hand derivatives. If

$$\lim_{h\to(0-0)}\frac{f(a-h)-f(a)}{h} \quad \text{or} \quad \lim_{x\to(a-0)}\frac{f(x)-f(a)}{x-a}$$

exists, then it is called the ***left hand derivative*** of f at a and is denoted by $f'(a-0)$ or $L\,f'(a)$.

Again, if

$$\lim_{h\to(0+0)}\frac{f(a-h)-f(a)}{h} \quad \text{or} \quad \lim_{x\to(a+0)}\frac{f(x)-f(a)}{x-a}$$

exists, then it is called the ***right hand derivative*** of f at a and is denoted by $f'(a+0)$ or $R\,f'(a)$.

Clearly, *a function f is differentiable at $x = a$ if $f'(a-0)$ and $f'(a+0)$ both exist and are equal.*

3.2 Continuity and differentiation. The relation between them is given by the theorem below.

Theorem. ***If a function is differentiable at a point, then it is continuous at that point but the converse is not necessarily true.***

[*Purvanchal* 04, *Lucknow* 06, *Avadh* 08, *Bilaspur* 09, *Kashi Vidyapeeth* 08, 10, *Bundelkhand* 08, 11]

Let $f(x)$ be differentiable at $x = a$, then $f'(x)$ exists and

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad (1)$$

We can write $f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a), \quad x \neq a.$

Taking limits on both sides as $x \to 0$, we get

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\,.\,\lim_{x\to a}(x-a)$$

$$= f'(a)\,.\,0 = 0. \qquad \text{[using (1)]}$$

i.e. $\lim_{x\to a} f(x) = f(a)$ which proves that f is necessarily continuous at $x = a$ if it is differentiable there.

We will now show that ***a continuous function need not be differentiable*** at a point.

Consider the modular function at the point $x = 0$ defined as follows :

$$f(x)=|x|=\begin{cases} x & \text{when } x>0 \\ 0 & \text{when } x=0 \\ -x & \text{when } x<0 \end{cases}$$

It is continuous at $x=0$ because limit of y as $x\to 0$ from the right is 0, and also the limit of y as $x\to 0$ from the left is 0.

Thus $\lim_{x\to 0} y$ exists and is equal to 0, which is also the value of y at $x=0$. Hence the function is continuous at $x=0$.

i.e. $\qquad f(0+0)=f(0-0)=f(0)$.

But at $x=0$,

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-|0|}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}=1.$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{h}=\lim_{h\to 0}\frac{|-h|-|0|}{(-h)}$$

$$=\lim_{h\to 0}\frac{h}{(-h)}=-1.$$

Since $Rf'(0)\neq Lf'(0)$, $f(x)$ is not differentiable at $x=0$.

Ex. 1. Test the continuity and differentiability of the function

$$f(x)=\begin{cases} 1+x & \text{if } x\le 2 \\ 5-x & \text{if } x\ge 2 \end{cases} \quad \text{at } x=2. \qquad \textit{[Avadh 2007]}$$

At $x=2$, the value of the function is $f(2)=5-2=3$.

Now $\quad f(2+0)=\lim_{h\to 0} f(2+h)=\lim_{h\to 0}[5-(2+h)]=3$.

$\quad f(2-0)=\lim_{h\to 0} f(2-h)=\lim_{h\to 0}[1+(2-h)]=3$.

Since $\quad f(2+0)=f(2-0)=f(2)$, f is continuous at $x=2$.

For differentiability of f at $x=2$, we have

$$R\,f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{[5-(2+h)]-3}{h}=\lim_{h\to 0}\frac{-h}{h}=-1$$

Again $\quad L\,f'(2)=\lim_{h\to 0}\dfrac{f(2-h)-f(2)}{-h}$

$$=\lim_{h\to 0}\frac{[1+(2-h)]-3}{-h}=\lim_{h\to 0}\frac{-h}{-h}=1.$$

Since $R\,f'(0)\neq L\,f'(0)$, f is not differentiable at $x=2$.

Ex. 2. Show that the function

$$f(x)=x\sin(1/x),\ x\neq 0;\ f(0)=0,$$

is continuous but not differentiable at $x=0$.

[***Jabalpur*** **07,** ***Avadh*** **08,** ***Indore*** **08,** ***Magadh*** **05, 09,** ***Bilaspur*** **10,** ***Rohilkhand*** **10,** ***Patna*** **09, 11,** ***Kashi Vidyapeeth*** **09, 12,** ***Raipur*** **11, 15]**

At $x=0$, we have, $\quad f(0+0)=\lim_{h\to 0} f(0+h)$

$$=\lim_{h\to 0}[h\sin(1/h)]=0\times \text{a finite number}=0$$

since $\lim_{h\to 0}\sin(1/h)$ oscillates between -1 and 1. (Ex. of § 2.4)

Again, $\quad f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}[(-h)\sin(-1/h)]$

$$=\lim_{h\to 0}[h\sin(1/h)]=0, \text{ as above}.$$

Hence, $f(0+0)=f(0-0)=f(0)$ which shows that the function f is continuous at $x=0$.

To test if f is differentiable at $x=0$, we have

$$R\,f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h\sin(1/h)-0}{h}=\lim_{h\to 0}\sin(1/h).$$

But $\lim_{h\to 0}\sin(1/h)$ oscillates between -1 and 1 and hence does not exist. Thus, $R\,f'(0)$ does not exist, Similarly,

$$L\,f'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{(-h)}=\lim_{h\to 0}\frac{(-h)\sin(-1/h)-0}{(-h)}$$

$$=-\lim_{h\to 0}\sin(1/h) \quad \text{does not exist}.$$

Hence, f is not differentiable at $x=0$.

Ex. 3. Show that the function

$$f(x) = x \tan^{-1} x \text{ when } x \neq 0 \text{ and } f(0) = 0$$

(i) is continuous but not differentiable at $x = 0$.

[*Kashi Vidyapeeth* 08, *Bundelkhand* 07, 12]

(ii) is not differentiable at $x = 0$.

[*Purvanchal* 08, *Kashi Vidyapeeth* 09, *Lucknow* 11]

At $x = 0$, we have, $f(0+0) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} [h \tan^{-1}(1/h)] = 0 \times \tfrac{1}{2}\pi = 0$$

since $\lim_{h \to 0} \tan^{-1}(1/h)$ tends to $\tan^{-1} \infty = \tfrac{1}{2}\pi$.

Again, $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} [(-h) \tan^{-1}(-1/h)]$

$$= \lim_{h \to 0} [h \tan^{-1}(1/h)] = 0, \text{ as above}.$$

Since $f(0+0) = f(0-0) = f(0)$, the function f is continuous at $x = 0$.

Now $R\,f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h \tan^{-1}(1/h) - 0}{h}$

$$= \lim_{h \to 0} \tan^{-1}(1/h) = \tfrac{1}{2}\pi.$$

$$L\,f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{(-h) \tan^{-1}(-1/h) - 0}{(-h)}$$

$$= -\lim_{h \to 0} \tan^{-1}(1/h) = -\tfrac{1}{2}\pi.$$

Since $R\,f'(0) \neq L\,f'(0)$, f is not differentiable at $x = 0$.

Ex. 4. If $f(x) = x \cdot \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$, $x \neq 0$ and $f(0) = 0$,

show that f is continuous but not differentiable at $x = 0$.

[*Meerut* 05, *Delhi* 06, *Kumaon* 09, *Lucknow* 09, *Raipur* 11, *Kashi Vidyapeeth* 12, *Bundelkhand* 14]

To test if f is continuous at $x = 0$, we have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$

$$= \lim_{h \to 0} h \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = 0 \times \frac{1-0}{1+0} = 0.$$

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} (-h) \dfrac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$

$$= \lim_{h \to 0} (-h) \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = 0 \times \frac{0-1}{0+1} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function f is continuous at $x = 0$.

To test if f is differentiable at $x = 0$, we have

$$R\,f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1-0}{1+0} = 1.$$

$$L\,f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} = \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = \frac{0-1}{0+1} = -1.$$

Since $R\,f'(0) \neq L\,f'(0)$, f is not differentiable at $x = 0$.

EXAMPLES

1. If $f(x) = |x| + |x-1|$, test for differentiability at $x = 0$.

[*Allahabad* 08, *Kashi Vidyapeeth* 10]

OR Show that the function $f(x) = |x| + |x-1|$, is not differentiable at $x = 0$ and $x = 1$. [*Delhi* 07 (2nd part), *Bundelkhand* 13]

2. Discuss the derivability of the function

$$f(x) = \begin{cases} x, & x < 0 \\ 2 - x, & 1 \le x \le 2 \\ -2 + 3x - x^2, & x > 2 \end{cases} \quad \text{at } x = 1, 2$$

[*Lucknow* 2012]

3. Prove that the function $f(x) = \begin{cases} x^2 - 1, & x \ge 1 \\ 1 - x, & x < 1 \end{cases}$

is not differentiable at $x = 1$. [*Indore* 06, *Jabalpur* 08]

4. Test the continuity and differentiability of the function

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 1, & x = 1 \end{cases} \quad \text{at } x = 0.$$

5. Prove that at $x=5$, the function $f(x)=|x-5|$ is continuous but not differentiable. [*Bilaspur* 2007]

6. Show that the following function is continuous and differentiable at $x=0$:

$$f(x)=\begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x=0. \end{cases}$$

[*Avadh* 06, *Bundelkhand* 06, *Bilaspur* 07, *Kumaon* 12]

7. Prove that the following function is differentiable at $x=0$:

$$f(x)=\begin{cases} e^{-1/x^2} \cdot \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x=0. \end{cases}$$

[*Avadh* 2006]

8. Show that the function defined as follows is everywhere continuous but has no differential coefficient at the origin :

$f(x)=x\{1+\frac{1}{3}\sin(\log x^2)\}$, when $x \neq 0$ and $f(0)=0$.

Test the function for continuity and differentiability at $x=0$.

[*Indore* 2007]

9. Test the continuity and differentiability of the following function at $x=0$ and $x=\frac{1}{2}\pi$:

$$f(x)=\begin{cases} 1, & \text{for } -\infty<x<0 \\ 1+\sin x, & \text{for } 0 \leq x<\frac{1}{2}\pi \\ 2+(x-\frac{1}{2}\pi)^2, & \text{for } \frac{1}{2}\pi \leq x<\infty. \end{cases}$$

[*Avadh* 09 (2nd part), *Rohilkhand* 11]

10. If $f(x)=\begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x=0, \end{cases}$ then show that $f(x)$ is continuous but not differentiable at $x=0$. [*Bilaspur* 08, *Raipur* 09]

11. Show that the following function $f(x)$ is continuous at $x=0$ but not differentiable at this point : $f(x)=\begin{cases} \dfrac{x e^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0, & x=0. \end{cases}$

[*Rohilkhand* 06, *Lucknow* 06, *Allahabad* 10 (1st part), *Raipur* 10, 12, 13, 14]

ANSWERS

2. Derivable at $x=2$ but not at $x=1$.
4. Neither continuous, nor differentiable.
9. Continuous for all x but differentiable for all x except at $x=0$.

3.3. Differential coefficient of the product of a constant and a function. Let k be a constant and $f(x)$ be a differentiable function. Then

$$(kf)'(x)=\lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}$$

$$=\lim_{h\to 0}\left\{k\,.\,\frac{f(x+h)-f(x)}{h}\right\}=kf'(x).$$

3.31. Differential coefficient of the sum or difference of two functions. Let f and g be two differentiable functions, then $(f \pm g)'(x)=f'(x) \pm g'(x)$.

Now $$(f \pm g)'(x)=\lim_{h\to 0}\frac{(f \pm g)(x+h)-(f \pm g)(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=f'(x) \pm g'(x).$$

3.32. Differential coefficient of the product of two functions. For two differentiable functions f and g,

$$(f g)'(x)=f(x)\,g'(x)+f'(x)\,g(x).$$

Let $F(x)=f(x)\,g(x)$,

Then $F(x+h)=f(x+h)\,g(x+h)$.

Therefore $F'(x)=(fg)'(x)$

$$=\lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)\{g(x+h)-g(x)\}+g(x)\{f(x+h)-f(x)\}}{h}$$

(by adding and subtracting one term in the numerator)

$$= \lim_{h \to 0} \left\{ f(x+h) \frac{g(x+h)-g(x)}{h} + g(x) \frac{f(x+h)-f(x)}{h} \right\}$$

$$= f(x)\, g'(x) + f'(x)\, g(x).$$

i.e. *the differential coefficient of the product of two functions*

= ***first function × diff. coeff. of second***

+ second function × diff. coeff. of first.

3.33. Differential coefficient of the quotient of two functions. Let f and g be two differentiable functions, then

$$(f/g)'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{\{g(x)\}^2}.$$

Let $F(x) = f(x)/g(x)$

Then $F(x+h) = f(x+h)/g(x+h)$

Therefore $F'(x) = (f/g)'(x)$

$$= \lim_{h \to 0} \frac{f(x+h)/g(x+h) - f(x)/g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x) - g(x+h)\, f(x)}{g(x+h)\, g(x) . h}$$

$$= \lim_{h \to 0} \frac{g(x)\left\{\dfrac{f(x+h)-f(x)}{h}\right\} - f(x)\left\{\dfrac{g(x+h)-g(x)}{h}\right\}}{g(x+h)\, g(x)}$$

(by adding and subtracting a term in the numerator, and then dividing both numerator and denominator by h)

$$= \frac{g(x)\, f'(x) - f(x)\, g'(x)}{\{g(x)\}^2}.$$

i.e., *the differential coefficient of the quotient of two functions*

$$= \frac{\textit{(Diff. coeff. of Numer.)(Denomr.)} - \textit{(Numer.)(Diff. coeff. of Denomr.)}}{\textit{Square of Denominator}}.$$

3.4. Table of standard results to be committed to memory. It is assumed that the student would have studied the proofs of differentiation of these simple functions in school.

1. $\frac{d}{dx}(k) = 0 \ (k = \text{constant})$
2. $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$
3. $\frac{d}{dx} e^x = e^x$
4. $\frac{d}{dx} \log x = \frac{1}{x}$
5. $\frac{d}{dx}(a^x) = a^x \log a$
6. $\frac{d}{dx} \log_a x = \frac{1}{x \log a}$
7. $\frac{d}{dx} \sin x = \cos x$
8. $\frac{d}{dx} \cos x = -\sin x$
9. $\frac{d}{dx} \tan x = \sec^2 x$
10. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$
11. $\frac{d}{dx} \sec x = \sec x . \tan x$
12. $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x . \cot x$

3.5. Chain rule of differentiation. This rule is useful in finding the derivative of a composite function, i.e. the function of a function.

Consider the function $\sin x^2$. This is certainly a function of x but its derivative cannot be found by the rules given so far. We know the derivative of x^2, and also of $\sin x$, but not of $\sin x^2$. In order to find the differential coefficient of $\sin x^2$, it is convenient to regard it as a function of x^2, which is itself a function of x. Thus $\sin x^2$ is regarded as a function of a function of x.

Consider the composite function

$$f(x) = (f_1 \circ f_2)(x) = f_1[f_2(x)]$$

where $f(x) = f_1(t)$ and $t = f_2(x)$ such that the range of the function f_2 is a subset of the domain of the function f_1. Let f, f_1 and f_2 all be differentiable functions.

Since $f(x) = f_1[f_2(x)]$

therefore $f(x+h) = f_1\{f_2(x+h)\}$

Also let $f_2(x+h)=t+k$ (1)

where h is the incremental change in x and k, the corresponding change in t.

Then, as $h \to 0$, $f_2(x+h) \to f_2(x)$ because $f_2(x)$ is a continuous function of x (being differentiable), i.e. $t+k \to t$. Hence as $h \to 0$, k also $\to 0$. Also from (1),

$$k = f_2(x+h)-t = f_2(x+h)-f_2(x) \qquad (2)$$

Since f is a differentiable function, therefore

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f_1\{f_2(x+h)\}-f_1\{f_2(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{f_1(t+k)-f_1(t)}{h}$$

$$= \lim_{h\to 0}\left\{\frac{f_1(t+k)-f_1(t)}{k}\cdot\frac{k}{h}\right\}$$

$$= \lim_{k\to 0}\frac{f_1(t+k)-f_1(t)}{k}.\lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h}$$

[using (2)]

$$= \frac{d}{dt}f_1(t).\frac{d}{dx}f_2(x).$$

i.e. $$\frac{d}{dx}f_1(t) = \frac{d}{dt}f_1(t).\frac{dt}{dx}.$$ [since $t=f_2(x)$]

This important theorem is usually remembered (after dropping the suffix of f_1) in the form

$$\frac{df}{dx} = \frac{df}{dt}.\frac{dt}{dx}. \qquad (3)$$

The same result (3) can be written in an alternative form

as $$\frac{d}{dx}f_1\{f_2(x)\} = \frac{df_1\{f_2(x)\}}{d\{f_2(x)\}}.\frac{df_2(x)}{dx}$$

or $$(f_1 \circ f_2)'(x) = f_1'(t).f_2'(x)$$

since $f(x)=(f_1 \circ f_2)(x)=f_1[f_2(x)]$ and $t=f_2(x)$.

In the case of a composite of more than two functions, we can generalise the chain rule of differentiation.

Ex. 1. Find the differential coefficient of $\sin x^2$.

Put $x^2=t$. Then

$$\frac{d}{dx}(\sin x^2) = \frac{d}{dx}(\sin t) = \frac{d}{dt}(\sin t)\frac{dt}{dx}$$

$$= \cos t.\frac{d(x^2)}{dx} = (\cos x^2)\,2x.$$

Ex. 2. Find the differential coefficient of $(ax^2+b)^3$.

Put $ax^2+b=t$. Then

$$\frac{d}{dx}(ax+b)^3 = \frac{d}{dx}(t^3) = \frac{d}{dt}(t^3)\frac{dt}{dx} = 3t^2\frac{d}{dx}(ax^2+b)$$

$$= 3(ax^2+b)^2.(2ax).$$

Ex. 3. Find differential coefficient of $\log\sin x$.

Put $\sin x=t$. Then

$$\frac{d}{dx}(\log\sin x) = \frac{d}{dx}(\log t) = \frac{d}{dt}(\log t)\frac{dt}{dx}$$

$$= \frac{1}{t}.\frac{d}{dx}\sin x = \frac{1}{\sin x}.\cos x = \cot x.$$

3.51. Differential coefficient of an inverse function. If there is a relation between x and y, we can either express y as a function of x, or x as a function of y. One of these functions, according to convenience, is called an ***inverse*** function. $\sin^{-1}x$, $\cos^{-1}x$ etc., are examples of such functions.

Let there be an inverse function $f^{-1}(x)$, i.e. let

$y=f^{-1}(x)$, when solved for x becomes $x=f(y)$.

In the former function, x is regarded as the independent variable and in the latter y. If both the function f and its inverse are differentiable, then dy/dx and dx/dy exist. The relation between these differential coefficients can be obtained as follows :

By the chain rule of differentiation, we have

$$\frac{dx}{dy}\times\frac{dy}{dx}=\frac{dx}{dx}=1 \qquad (1)$$

which shows that dy/dx and dx/dy are reciprocals of each other. Therefore, from (1), provided $dx/dy \neq 0$,

$$\frac{dy}{dx}=1\bigg/\frac{dx}{dy}.$$

This formula is useful in differentiating inverse functions.

3.6. Inverse Trigonometric Functions.

3.61. Differential Coefficient of $\sin^{-1} x$.

Let $\sin^{-1} x = y$ then $x=\sin y$ or $dx/dy=\cos y$.

Using the formula of the previous article, we have

$$\frac{dy}{dx}=1\bigg/\frac{dx}{dy}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

i.e.
$$\frac{d}{dx}\sin^{-1} x=\frac{1}{\sqrt{1-x^2}}.$$

It should be carefully noted that whereas $\sin^2 x$ means $(\sin x)^2$, $\sin^{-1} x$ does not mean $(\sin x)^{-1}$.

3.62. Differential Coefficient of $\cos^{-1} x$. This we can find by a procedure similar to the above, or as follows:

By trigonometry, we have $\cos^{-1} x=\frac{1}{2}\pi-\sin^{-1} x$.

Differentiating both sides, we get at once

$$\frac{d}{dx}\cos^{-1} x=-\frac{1}{\sqrt{1-x^2}}$$

3.63. Differential coefficients of $\tan^{-1} x$ and $\cot^{-1} x$.

Let $y=\tan^{-1} x$ then $\tan y = x$ or $dx/dy=\sec^2 y$.

Now
$$\frac{dy}{dx}=1\bigg/\frac{dx}{dy}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}$$

i.e.
$$\frac{d}{dx}\tan^{-1} x=\frac{1}{1+x^2}.$$

We have similarly, $\dfrac{d}{dx}\cot^{-1} x=-\dfrac{1}{1+x^2}$.

This also follows immediately from the equation

$$\cot^{-1} x=\tfrac{1}{2}\pi-\tan^{-1} x.$$

3.64. Differential Coefficients of $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$.

Let $y=\sec^{-1} x$ then $\sec y = x$.

Now
$$\frac{dy}{dx}=1\bigg/\frac{dx}{dy}=\frac{1}{\sec y\tan y}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}.$$

i.e.
$$\frac{d}{dx}\sec^{-1} x=\frac{1}{x\sqrt{x^2-1}}.$$

Similarly
$$\frac{d}{dx}\operatorname{cosec}^{-1} x=-\frac{1}{x\sqrt{x^2-1}}.$$

This also follows immediately from the equation

$$\operatorname{cosec}^{-1} x=\tfrac{1}{2}\pi-\sec^{-1} x.$$

To summarise :

1. $\dfrac{d}{dx}\sin^{-1} x=\dfrac{1}{\sqrt{1-x^2}}$ 2. $\dfrac{d}{dx}\cos^{-1} x=-\dfrac{1}{\sqrt{1-x^2}}$

3. $\dfrac{d}{dx}\tan^{-1} x=\dfrac{1}{1+x^2}$ 4. $\dfrac{d}{dx}\cot^{-1} x=-\dfrac{1}{1+x^2}$

5. $\dfrac{d}{dx}\sec^{-1} x=\dfrac{1}{x\sqrt{x^2-1}}$ 6. $\dfrac{d}{dx}\operatorname{cosec}^{-1} x=-\dfrac{1}{x\sqrt{x^2-1}}$

3.7. Hyperbolic Functions. These functions are defined as follows:

$$\cosh x=\tfrac{1}{2}(e^x+e^{-x}) \qquad \sinh x=\tfrac{1}{2}(e^x-e^{-x})$$
$$\tanh x=\sinh x/\cosh x \qquad \coth x=\cosh x/\sinh x$$
$$\operatorname{sech} x=1/\cosh x \qquad \operatorname{cosech} x=1/\sinh x.$$

(i) We notice that

$$\cosh^2 x-\sinh^2 x=\tfrac{1}{4}(e^x+e^{-x})^2-\tfrac{1}{4}(e^x+e^{-x})^2$$
$$=\tfrac{1}{4}(e^{2x}+2+e^{-2x})-\tfrac{1}{4}(e^{2x}-2+e^{-2x})=\tfrac{1}{4}(2+2);$$

$\therefore$

i.e. $$\cosh^2 x - \sinh^2 x = 1 .$$

(ii) Again $$\frac{d}{dx}\sinh x = \tfrac{1}{2}\frac{d}{dx}(e^x - e^{-x}) = \tfrac{1}{2}(e^x + e^{-x});$$

i.e. $$\frac{d}{dx}\sinh x = \cosh x$$

Similarly $$\frac{d}{dx}\cosh x = \sinh x$$

(iii) Other formulae, which can be easily verified, are :

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$$

3.71. Inverse Hyperbolic Functions. These functions are defined in a similar manner as inverse trigonometric functions

If $y = \cosh^{-1} x$, then $x = \cosh y = \frac{1}{2}(e^{y} + e^{-y})$.

Multiplying by $2e^{y}$, and transposing, $e^{2y} - 2xe^{y} + 1 = 0$.

Solving as a quadratic in e^{y},

$$e^{y} = x \pm \sqrt{x^2 - 1}.$$

Therefore $$y = \log\{x \pm \sqrt{x^2 - 1}\}.$$

The principal value is defined as the one in which the positive sign is taken, and is the value which is usually denoted by $\cosh^{-1} x$.

Hence $$\cosh^{-1} x = \log\{x + \sqrt{x^2 - 1}\}$$

Similarly $$\sinh^{-1} x = \log\{x + \sqrt{x^2 + 1}\}$$

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1-x}{1+x}.$$

The differentiation of the inverse hyperbolic functions, therefore, would present no difficulties.

3.8. Differentiation with regard to a function. Suppose we have to differentiate $\sin x^2$ with regard to x^2. This means that we consider x^2 to be the independent variable. Putting $x^2 = t$, we have diff. coeff. of $\sin x^2$ with regard to x^2

$$= \frac{d\,(\sin x^2)}{d\,(x^2)} = \frac{d}{dt}(\sin t) = \cos t = \cos x^2.$$

We can derive a general rule as follows :

Diff. coeff. of $f(x)$ w.r.t. $\phi(x)$, i.e. $\dfrac{df(x)}{d\phi(x)}$

$$= \frac{df(x)}{dt}, \text{ where } t = \phi(x)$$

$$= \frac{df(x)}{dx}\cdot\frac{dx}{dt}, = \frac{df(x)}{dx}\Big/\frac{dt}{dx} \quad \text{by § 3.51.}$$

Hence $$\frac{df(x)}{d\phi(x)} = \frac{df(x)}{dx}\Big/\frac{d\phi(x)}{dx}.$$

Ex. Differentiate $\sin x$ w.r.t. $\log x$.

Thus, diff. coeff. of $\sin x$ w.r.t. $\log x$

$$= \frac{d\,(\sin x)}{d\,(\log x)} = \left(\frac{d}{dx}\sin x\right)\Big/\frac{d}{dx}\log x = x\cos x.$$

3.81. Generalisation of the rule for differentiating a function of a function. To differentiate the function of a function, in practice actual substitution can be dispensed with. A help towards this end is to note that the Chain Rule formula

$$\frac{df(t)}{dx} = \frac{df}{dt}\cdot\frac{dt}{dx} \quad \text{can also be written as follows :}$$

$$\frac{d}{dx}f_1\{f_2(x)\} = \frac{df_1\{f_2(x)\}}{d\{f_2(x)\}}\cdot\frac{df_2(x)}{dx}.$$

Ex. Differentiate $\log(\sin x^2)^n$ with regard to x.

We first differentiate this with regard to $\log(\sin x^2)$ and get $n(\log \sin x^2)^{n-1}$. We then differentiate $\log(\sin x^2)$ w.r.t. $\sin x^2$ and get $1/\sin x^2$.

We have to multiply both these results by the differential coefficient of $\sin x^2$ with regard to x^2. For that we differentiate it first with regard to x^2 and thus get $\cos x^2$. We have now to multiply the product of all these by the differential coefficient of x^2 with regard to x. The latter is $2x$. Hence the required differential coefficient is

$$\frac{d}{dx}(\log \sin x^2)^n = \frac{d(\log \sin x^2)^n}{d(\log \sin x^2)} \cdot \frac{d(\log \sin x^2)}{d(\sin x^2)} \cdot \frac{d(\sin x^2)}{d(x^2)} \cdot \frac{d(x^2)}{dx}$$

$$= n(\log \sin x^2)^{n-1} . (1/\sin x^2) . \cos x^2 . 2x .$$

This is an example of differentiation of a function of a function very many times.

3.82. Logarithmic Differentiation. When a function of x is raised to a power which is again a function of x, neither the formula for x^n nor that for a^x is applicable. Then we have to take logarithms and differentiate. This process, called ***logarithmic differentiation***, is also useful when the function consists of the product of a number of functions.

Ex. 1. Find the differential coefficient of $(\sin x)^{\log x}$.

Let $y = (\sin x)^{\log x}$.

Taking logs, $\log y = \log x . \log \sin x$.

Differentiating, $\frac{1}{y} . \frac{dy}{dx} = \log x . \frac{1}{\sin x} . \cos x + \frac{1}{x} \log \sin x$.

Therefore $\frac{dy}{dx} = (\sin x)^{\log x} \{\log x . \cot x + (\log \sin x)/x\}$

Ex. 2. Find the differential coefficient of $x^x + (\sin x)^{\log x}$.

Here we cannot take logarithms directly. In such cases we have to ***find the differential coefficient of each term separately.***

Let $y = x^x$. Then $\log y = x \log x$.

Therefore $\frac{1}{y} . \frac{dy}{dx} = x . \frac{1}{x} + \log x$ or $\frac{dy}{dx} = x^x (1 + \log x)$.

Again let $z = (\sin x)^{\log x}$. Then $\log z = \log x . \log \sin x$

So $\frac{dz}{dx} = (\sin x)^{\log x} \{\log x . \cot x + (\log \sin x)/x\}$

Therefore $\frac{d}{dx}\{x^x + (\sin x)^{\log x}\}$

$$= x^x (1 + \log x) + (\sin x)^{\log x} \{\log x . \cot x + (\log \sin x)/x\}.$$

3.83. Differential Coefficient of the product of any number of functions.

Let $y = f_1(x) . f_2(x) . f_3(x) \ldots f_n(x)$.

Then $\log y = \log f_1(x) + \log f_2(x) + \ldots + \log f_n(x)$.

Therefore $\frac{1}{y} . \frac{dy}{dx} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots + \frac{f_n'(x)}{f_n(x)}$

or $\frac{dy}{dx} = f_1'(x) f_2(x) f_3(x) \ldots f_n(x) + f_1(x) f_2'(x) f_3(x) \ldots f_n(x) + \ldots$

i.e. to differentiate the product of any number of functions multiply the differential coefficient of each function taken separately by the product of all the remaining functions and add up the results.

EXAMPLES

Find the differential coefficients of the following :

1. $a^{\sin^{-1} x}$ with respect to $\sin^{-1} x$.
2. $\log_{10} x$ with respect to x^2.
3. $\tan^{-1}\{(\sqrt{1+x^2} - 1)/x\}$ with respect to $\tan^{-1} x$.
4. $\tan^{-1}\{2x/(1-x^2)\}$ with respect to $\sin^{-1}\{2x/(1+x^2)\}$.
5. $(\log x)^{\sin x}$.
6. $x \log x . \log \log x$.
7. $(\tan x)^{\cot x} + (\cot x)^{\tan x}$.
8. e^{x^x}.
9. $\log \frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1}$.
10. $\frac{x^2 \sqrt{x^2 + 4}}{\sqrt{x^2 + 3}}$.
11. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

ANSWERS

1. $a^{\sin^{-1} x} \log a$. **2.** $\frac{1}{2}(\log_{10} e)/x^2$. **3.** $\frac{1}{2}$. **4.** 1.

5. $(\log x)^{\sin x} \{(\log x)/(x \log x) + \cos x \log \log x\}$.

6. $\log x . \log \log x + \log \log x + 1$.

7. $(\tan x)^{\cot x} \operatorname{cosec}^2 x \log(e \cot x) - (\cot x)^{\tan x} \sec^2 x \log(e \tan x)$.

8. $x^x e^{x^x} \log(ex)$.

9. $1/x\sqrt{x+1}$.

10. $x^2(x^2+4)^{1/2}(x^2+3)^{-1/2}[(2/x)-x/(x^2+4)(x^2+3)]$.

3.84. Implicit functions. If the relation between x and y is given by an equation involving both, and this equation is not immediately solvable for y, then y is called an ***implicit function*** of x. On the other hand, when y is given in terms of x, y is called an ***explicit function*** of x.

In the case of implicit functions, there might be more than one value of y for each value of x, and there might be other complications. But, without paying heed to these, it is possible to get the value of dy/dx by mere differentiation of the given equation as it stands.

Ex. 1. Find dy/dx when $x^5+y^5+5xy-c=0$.

Differentiating, we get $5x^4+5y^4y'+5(xy'+y)=0$.

Therefore $\quad dy/dx=-(x^4+y)/(x+y^4)$.

Ex. 2. Find dy/dx if $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\dots\dots\infty}}}$.

We have $\quad y=\sqrt{\sin x+y} \quad$ or $\quad y^2-y=\sin x$. $\quad$ (on squaring)

Differentiating, we get $\quad (2y-1)\dfrac{dy}{dx}=\cos x \quad$ or $\quad \dfrac{dy}{dx}=\dfrac{\cos x}{2y-1}$.

3.85. Parametric equations. Sometimes x and y are both expressed in terms of a third variable, usually called a ***parameter***. We can always find dy/dx in such cases, without first eliminating the parameter.

Thus, if $x=f_1(t)$, and $y=f_2(t)$, then

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}.$$

Ex. If $x=a(t-\sin t)$ and $y=a(1-\cos t)$, find dy/dx.

Differentiating, $\quad dx/dt=a(1-\cos t)$ and $dy/dt=a\sin t$.

Then $\quad \dfrac{dy}{dx}=\dfrac{dy}{dt}\Big/\dfrac{dx}{dt}=\dfrac{a\sin t}{a(1-\cos t)}=\dfrac{2\sin\frac{1}{2}t\cos\frac{1}{2}t}{2\sin^2\frac{1}{2}t}=\cot\frac{1}{2}t$

3.86. Transformation. Sometimes an algebraic or trigonometrical transformation before differentiation materially shortens the work.

Ex. 1. Differentiate $\tan^{-1}\{2x/(1-x^2)\}$

Put $x=\tan\theta$. $\quad$ Then $\quad \dfrac{2x}{1-x^2}=\dfrac{2\tan\theta}{1-\tan^2\theta}=\tan 2\theta$

Therefore $\quad \dfrac{d}{dx}\tan^{-1}\{2x/(1-x^2)\}=\dfrac{d}{dx}\tan^{-1}(\tan 2\theta)$

$$=\frac{d}{dx}(2\theta)=\frac{d}{dx}(2\tan^{-1}x)=\frac{2}{1+x^2}.$$

Ex. 2. If $y\sqrt{1-x^2}+x\sqrt{1-y^2}=1$, find dy/dx.

Let $\quad x=\sin A \quad$ and $\quad y=\sin B$, $\quad$ then

$$\sin B\sqrt{1-\sin^2 A}+\sin A\sqrt{1-\sin^2 B}=1$$

or $\quad \sin B\cos A+\sin A\cos B=1 \quad$ i.e. $\quad \sin(A+B)=1$

or $\quad A+B=\sin^{-1}1 \quad$ i.e. $\quad \sin^{-1}x+\sin^{-1}y=\sin^{-1}1$.

Differentiating, we get

$$\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=0 \quad \text{or} \quad \frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}.$$

Ex. 3. Differentiate $\dfrac{x^5+3x^4+6x^3+2x^2+6x+2}{x^3+2}$.

Dividing by the denominator, we find that the given expression

$$=x^2+3x+6-10/(x^3+2).$$

Differentiating, we find that the required differential coefficient

$$=2x+3+\frac{10.3x^2}{(x^3+2)^2}.$$

Ex. 4. Differentiate $\log\dfrac{(x^2-1)^{1/3}x^{2/3}}{ax+b}$.

The given expression $=\frac{1}{3}\log(x^2-1)+\frac{2}{3}\log x-\log(ax+b)$.

Therefore the required diff. coeff. $=\dfrac{2x}{3(x^2-1)}+\dfrac{2}{3x}-\dfrac{a}{ax+b}$.

EXAMPLES

Find dy/dx when

1. $x^y \cdot y^x = 1$. 2. $x^y + y^x = a^b$.

3. $e^x \log y = \sin^{-1} x + \sin^{-1} y$.

4. If $\sin y = x \sin(a+y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

[HINT: Write $x = \sin y / \sin(a+y)$, then find $dy/dx = 1/(dx/dy)$]

5. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $dy/dx = -1/(1+x)^2$.

[HINT: Transpose, square and solve for y. Reject the root $y = x$ which does not satisfy the given equation and differentiate the other root.]

Find dy/dx when

6. $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots \text{to inf}}}$. 7. $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cos x \ldots\ldots \infty}}}$.

8. $x = a(\cos t + \log \tan \tfrac{1}{2}t)$, $y = a \sin t$.

9. $x = 3\cos t - 2\cos^3 t$, $y = 3\sin t - 2\sin^3 t$.

10. $x = 3at/(1+t^3)$, $y = 3at^2/(1+t^3)$.

11. $x = \log t + \sin t$, $y = e' + \cos t$.

Differentiate the following :

12. $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$. 13. $\tan^{-1} \dfrac{x}{\sqrt{a^2 - x^2}}$. [Put $x = a \sin \theta$]

14. $\tan^{-1}\left(\dfrac{a-x}{1+ax}\right)$. [Use $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$]

15. $\tan^{-1} \dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$. [Put $x^2 = \cos 2\theta$ and show that $y = \tfrac{1}{4}\pi + \theta$. Now differentiate.]

ANSWERS

1. $-y(y + x\log y)/x(x + y\log x)$.

2. $-[yx^{y-1} + y^x \log y]/[x^y \log x + xy^{x-1}]$.

3. $y\sqrt{1-y^2}\{1 - \sqrt{1-x^2}\, e^x \log y\}/\{e^x\sqrt{1-y^2} - y\}\sqrt{1-x^2}$.

6. $\dfrac{-(y^2 \tan x)}{1 - y \log \cos x}$. **7.** $\dfrac{(1+y)\cos x + y \sin x}{1 + 2y + \cos x - \sin x}$.

8. $\tan t$. **9.** $\cot t$. **10.** $t(2-t^3)/(1-2t^3)$.

11. $t(e' - \sin t)/(1 + t\cos t)$. **12.** $-\tfrac{1}{2}$.

13. $1/\sqrt{a^2 - x^2}$. **14.** $-1/(1+x^2)$. **15.** $3/\sqrt{1-x^2}$.

3.9. Differentiation from first principles. Differentiation from first principles means that the propositions about the differentiation of sums, products, function of functions, etc., are not to be applied; neither are the results of differentiating the standard forms to be assumed. The process is of no practical utility; but a knowledge of it is desirable. So those standard forms which have not already been differentiated from first principles, and also some typical examples, are considered below.

(i) $$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x \cdot \frac{e^h - 1}{h} = \lim_{h \to 0} e^x \cdot \frac{\left[1 + h + \dfrac{h^2}{2!} + \dfrac{h^3}{3!} + \cdots\right] - 1}{h}$$

$$= e^x \lim_{h \to 0} \left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right] = e^x \cdot 1 = e^x.$$

(ii) $$\frac{d}{dx} \sec x = \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h\cos(x+h)\cos x} = \lim_{h \to 0} \frac{2\sin(x + \tfrac{1}{2}h)\sin\tfrac{1}{2}h}{h\cos(x+h)\cos x}$$

$$= \lim_{h \to 0} \left\{\frac{\sin(x + \tfrac{1}{2}h)}{h\cos(x+h)\cos x} \cdot \frac{\sin\tfrac{1}{2}h}{\tfrac{1}{2}h}\right\} = \tan x \sec x.$$

The differentiation of $\operatorname{cosec} x$ is similar.

(iii) To differentiate $\sin^{-1} x$, put $y = \sin^{-1} x$ and $y + k = \sin^{-1}(x+h)$. As $h \to 0$, k also $\to 0$. We have now $x + h = \sin(y+k)$ and $x = \sin y$.

Therefore $$h = \sin(y+k) - \sin y,$$

or $$1 = \frac{\sin(y+k)-\sin y}{k} \cdot \frac{k}{h}$$

$$= \frac{\sin(y+k)-\sin y}{k} \cdot \frac{\sin^{-1}(x+h)-\sin^{-1} x}{h}$$

Taking limits as $h \to 0$, $$1 = \frac{d \sin y}{dy} \cdot \frac{d \sin^{-1} x}{dx} = \cos y \frac{d \sin^{-1} x}{dx}.$$

Therefore $$\frac{d}{dx} \sin^{-1} x = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

The above method will apply in the case of all the remaining inverse circular functions.

Ex. Differentiate $\sin x^2$ from first principles.

$$\frac{d}{dx}\sin x^2 = \lim_{h \to 0} \frac{\sin(x+h)^2 - \sin x^2}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{\sin(x+h)^2 - \sin x^2}{(x+h)^2 - x} \cdot \frac{(x+h)^2 - x^2}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{2\cos\frac{1}{2}\{(x+h)^2 + x^2\}\sin\frac{1}{2}\{(x+h)^2 - x^2\}}{2hx + h^2} \frac{2hx + h^2}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \cos\tfrac{1}{2}\{(x+h)^2 + x^2\} \frac{\sin\frac{1}{2}(2hx+h^2)}{\frac{1}{2}(2hx+h^2)} (2x+h) \right\}$$

$$= (\cos x^2)\, 2x.$$

because as $h \to 0$, $$\frac{\sin\frac{1}{2}(2hx+h^2)}{\frac{1}{2}(2hx+h^2)} \to 1; \quad 2x + h \to 2x;$$

and $$\cos\tfrac{1}{2}\{(x+h)^2 + x^2\} \to \cos\tfrac{1}{2}\{x^2 + x^2\}.$$

3.10. Transformation of equations. It is sometimes required to change the independent variable from one to another in a differential equation to simplify it using the Chain Rule of differentiation. This is illustrated below with an example.

Ex. Transform the equation

$$(1+x^2)^2 \frac{d^2 y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + y = 0$$

by the substitution $x = \tan z$. [*Indore* 07, *Raipur* 15]

We have to change the independent variable from x to z.

Now $x = \tan z$, so $$\frac{dz}{dx} = 1\Big/\frac{dx}{dz} = \frac{1}{\sec^2 z} = \cos^2 z.$$

Hence $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \cos^2 z \frac{dy}{dz}. \qquad (1)$$

and $$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dx}\right)\cdot\frac{dz}{dx} = \frac{d}{dz}\left(\cos^2 z \frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$= \left[-2\cos z \sin z \frac{dy}{dz} + \cos^2 z \frac{d^2 y}{dz^2}\right](\cos^2 z)$$

or $$\frac{d^2 y}{dx^2} = -2\cos^3 z \sin z \frac{dy}{dz} + \cos^4 z \frac{d^2 y}{dz^2}. \qquad (2)$$

Substituting $x = \tan z$, (1) and (2) in the given equation, we have

$$(\sec^4 z)\left[-2\cos^3 z \sin z \frac{dy}{dz} + \cos^4 z \frac{d^2 y}{dz^2}\right] + 2\tan z \sec^2 z \cos^2 z \frac{dy}{dz} + y = 0$$

or $$\left[-2\tan z \frac{dy}{dz} + \frac{d^2 y}{dz^2}\right] + 2\tan z \frac{dy}{dz} + y = 0$$

or $$\frac{d^2 y}{dz^2} + y = 0$$ which is the transformed equation.

EXAMPLES

Transform the equations / change the independent variable in the following equations :

1. $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0$ by the substitution $x = e^z$.

[*Raipur* 12, 14]

2. $x^2 \frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + \frac{a^2}{x^2} y = 0$ given that $x = 1/z$.

3. $(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0$ where $x = \cos z$.

[*Jabalpur* 07, *Indore* 08]

4. $\cos x\dfrac{d^2y}{dx^2}+\sin x\dfrac{dy}{dx}-2y\cos^3 x=2\cos^5 x$ where $z=\sin x$.

[*Jabalpur* 2006]

5. $\sin^2 2z\dfrac{d^2y}{dz^2}+\sin 4z\dfrac{dy}{dz}+4y=0$ by putting $\tan z=e^x$.

[*Jabalpur* 06, *Bilaspur* 10]

6. $\dfrac{d^2y}{dx^2}+\cot x\dfrac{dy}{dx}+4y\operatorname{cosec}^2 x=0$ by putting $z=2\log\tan\frac{1}{2}x$.

[*Jabalpur* 06, *Raipur* 15]

ANSWERS

1. $\dfrac{d^2y}{dz^2}+y=0$.
2. $\dfrac{d^2y}{dz^2}+a^2y=0$.
3. $\dfrac{d^2y}{dz^2}+n^2y=0$.
4. $\dfrac{d^2y}{dz^2}-2y=2(1-z^2)$.
5. $\dfrac{d^2y}{dx^2}+y=0$.
6. $\dfrac{d^2y}{dz^2}+y=0$.

4

Simple Applications

4.1. Velocity. The student must be familiar with the idea of velocity. Suppose now that a particle is falling under gravity, so that the velocity of the particle goes on increasing. What exactly is meant by the velocity at a particular instant?

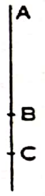
Assume that the particle started from A at the instant from which we reckon time.

Assume further that at time t the particle is at B, and let AB $= s$. What is the velocity at time t?

Suppose we wait till the time has become $t + h$. Let the particle then be at C, BC being equal to k.

Then k/h is the average velocity during the interval h under consideration.

But this average velocity must be greater than the velocity at the beginning of the interval because the velocity is increasing. If we take h to be very small, the velocity at time t would be very nearly equal to k/h but not exactly. This difficulty would remain, however small h may be taken, provided it is not made zero.



But if we make h zero, then k also becomes zero; so k/h takes the form 0/0 and we cannot infer what the velocity is.

Hence we have to define velocity as the limit of k/h when $h \to 0$.

Now, if we regard t as an independent variable, we may write $s = f(t)$.

Thus $$\textbf{velocity} = \frac{ds}{dt}$$

We may expect, therefore, that differential calculus would be very useful in mechanics.

4.11. Acceleration. If v is the velocity at time t, considerations similar to the above will lead us to define acceleration as follows :

$$\textbf{acceleration} = \frac{dv}{dt}$$

Ex. A particle is projected downwards with a velocity u and the acceleration is constant and equal to f. Find the velocity, at any instant and the space described in time t.

Here $$\frac{dv}{dt} = f$$

We can guess, therefore, that

$$v = ft + A \qquad (1)$$

where A is a constant.

[The student might wonder why $ft + A$ has been taken instead of simply ft; the latter value of v does make dv/dt equal to f as required. But he must remember that ft is merely a particular case of $ft + A$, viz., the case when A is zero; and a particular case might not serve our purpose.]

It is given that the velocity is equal to u when $t = 0$. Substituting these value of v and t in (1), we get

$$u = f\,.0 + A\,,$$

which gives the value of A. Thus (1) becomes

$$v = u + ft$$

But $v = ds/dt$. Hence

$$\frac{ds}{dt} = u + ft$$

We can guess, therefore, that

$$s = ut + \frac{1}{2}ft^2 + B$$

But $s = 0$ when $t = 0$. Hence $B = 0$. We have therefore

$$s = ut + \frac{1}{2}ft^2$$

4.2. Increments and their ratios. If x is 3 at first, and then becomes 4, we say that there is an *increment* of 1 in x. If, however, x is 3 at first and becomes 2 afterwards, there is a diminution in x; but it is more convenient to say that there is an increment of -1 in x.

In general, if x changes from x_1 to x_2, the increment in x is $x_2 - x_1$, whether $x_2 - x_1$ be positive or negative. The increment in x has so far been denoted by h, and the increment in y by k. But in the application of differential calculus it is more convenient to denote the increment in x by δx, the increment in y by δy, and so on.

The reason is that if there are many quantities which change and we use letters like $h, k, l, \ldots$ to denote their increments, it becomes difficult to remember which symbol is the increment of a particular quantity. But the student must be careful to regard δx as one symbol, and not as the product of δ and x.

We shall now show that ***the differential coefficient of a function y of x with respect to x is the limit of the ratio of corresponding increments in y and x, as the increment in x tends to zero*** (positive and negative increments both being considered).

If y is a function of x, say $y = f(x)$, and x, y and $x + h$, $y + k$ are pairs of corresponding values, then the ratio of the increment in y to that in x is k/h.

Now $$\lim_{h\to 0}\frac{k}{h} = \lim_{h\to 0}\frac{y+k-y}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

i.e.
$$\lim_{h\to 0}\frac{k}{h}=\frac{dy}{dx},$$

which proves the proposition.

In the usual notation for increments,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}.$$

4.3. Rate of increase. Consider the problem of § 4.1 again, in which s is the distance described by a particle in time t. The average rate of increase of s in an interval of time δt

$$=\frac{\text{increase in } s \text{ in the interval } \delta t}{\delta t}, \text{ i.e. } \frac{\delta s}{\delta t}.$$

Therefore, the rate of increase of s at time t

$$=\lim_{\delta t\to 0}\frac{\delta s}{\delta t}=\frac{ds}{dt} \quad \text{(by § 4.2).}$$

So we can assign a new meaning to the differential coefficient ds/dt. It is the rate at which s increases with t.

But it is not necessary that the independent variable be time. We can conceive of the rate (at a given instant) of growth of y with x. –

The differential coefficient dy/dx will then be the rate at which y increases with x.

Ex. A balloon, which always remains spherical, has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is 10 inches.

If the radius is x, the volume (say y) is $\frac{4}{3}\pi x^3$.

Hence the rate at which the volume increases with x is

$$\frac{d}{dx}\left(\tfrac{4}{3}\pi x^3\right) \text{ i.e. } \tfrac{4}{3}\pi . 3x^2.$$

Hence the required rate of increase $=\frac{4}{3}\pi . 3 . 10^2$ cu. in. per unit (1 in.) increase in the radius.

[*Verification.* We can verify the above by elementary methods. Suppose the radius becomes 10 inches + 0.01 inch.

Then the new volume is $\frac{4}{3}\pi(10.01)^3$ cu. in.

The old volume was 10^3 cu. in.

Hence the increase in the volume $=\frac{4}{3}\pi(10.01^3 - 10^3)$ cu. in.

The increase in radius = 0.01 in.

Thus the average rate of increase of volume with radius (in cubic inches per inch)

$$=\frac{4}{3}\pi.\frac{10.01^3-10^3}{0.01}$$

$$=\frac{4}{3}\pi\{10^3+3.10^2.(0.01)+3.10.(0.01)^2+$$
$$(0.01)^3-10^3\}/0.01, \text{ by the Binomial theorem,}$$

$$=\frac{4}{3}\pi\{3.10^2+3.10.(0.01)+(0.01)^2\}=\frac{4}{3}\pi.3.10^2 \text{ nearly.}$$

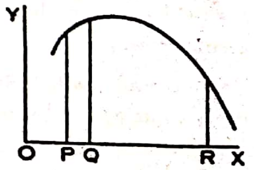
which is the same as before. Hence we have approximately verified the calculation made above.]

By supposing the increase in the radius to be h instead of 0.01, the average rate would have come out as

$$=\tfrac{4}{3}\pi\{3.10^2+3.10.h+h^2\}$$

and taking the limit as $h\to 0$, we can easily show that the exact (and not merely an approximate) rate is $\frac{4}{3}\pi . 3. 10^2$.

4.31. Meaning of the sign of the Differential Coefficient. Let y or $f(x)$ be a function of x. When $x=a$, the function has the value $f(a)$. If x is given a small positive increment, $f(x)$ might become greater than $f(a)$, or might not. If it does, we may express this fact by saying that $f(x)$ *increases with* x at $x=a$.



We took the increment in x to be *small*. If we take a large increment (say PR instead of PQ as shown in the figure), the new value of the function might be less than before even though the function may be increasing with x. When we say that $f(x)$ increases with x at $x = a$, we mean merely that we can find a number h_1 (i.e. PQ in the figure) such that $f(a+h) > f(a)$ when $0 < h < h_1$. This is sometimes expressed by saying that $f(a+h) > f(a)$ *for sufficiently small (positive) values of h.*

We can now show that ***if for any value, say a, of x, the differential coefficient dy/dx is positive, it means that y increases with x at x = a. On the other hand, if dy/dx is negative at x = a, it means that y decreases as x increases.***

For, if dy/dx is positive at $x = a$,

$$\frac{f(a+h)-f(a)}{h}$$

must be positive, at least for sufficiently small values of h. This means that $f(a+h) - f(a)$ must be positive when h is positive and negative when h is negative, provided $|\,h\,|$ is sufficiently small, i.e. $f(x)$ must increase with x at $x = a$.

Similarly, if dy/dx is negative at $x = a$, $f(x)$ must decrease as x increases.

The student should note that "$f(x)$ decreases with x" means that $f(x)$ decreases as x decreases, and so he must not use this expression when he means that "$f(x)$ decreases as x increases".

We shall show in a later chapter what happens when dy/dx is zero and is thus neither positive nor negative. We shall show that at such a point $f(x)$ has got, in general, either a maximum (greatest) or minimum (least) value.

Ex. 1. Show that the function x^3 steadily increases from $x = -\infty$ to $x = +\infty$ but x^4 decreases from $x = -\infty$ to $x = 0$ and then increases.

If $y = x^3$, $dy/dx = 3x^2$, which is always positive. Hence for every value of x, x^3 increases as x increases.

Again if $y = x^4$, $dy/dx = 4x^3$, which is negative when x is negative, and positive when x is positive, which shows that for negative values of x, x^4 diminishes as x increases (algebraically, of course) but for positive values of x, x^4 increases with x.

[The student should draw graphs of these functions.]

Ex. 2. The top of a ladder 20 feet long is resting against a vertical wall on a level pavement, when the ladder begins to slide outwards. At the moment when the foot of the ladder is 12 feet from the wall, it is sliding away from the wall at the rate of 2 feet per second. How fast is the top sliding downwards at this instant? How far is the foot from the wall when it and the top are moving at the same rate?

At time t seconds reckoned from some fixed instant, let the distance of the foot of the ladder from the wall be x feet and the height from the pavement to the top of the ladder be y feet. Then, by Pythagoras theorem,

$$x^2 + y^2 = 20^2. \tag{1}$$

Differentiating with respect to t, we have

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0. \tag{2}$$

At the moment mentioned in the problem, $x = 12$, $dx/dt = 2$. So $y = 16$ by (1), and $dy/dt = -\frac{3}{2}$ by (2).

Hence the top of the ladder is sliding downwards at the rate of $1\frac{1}{2}$ feet per second.

Again, if dx/dt and dy/dt and are equal numerically, (2) shows that x must be equal to y. Hence by (1), the value of x then is $10\sqrt{2}$; i.e., the foot of the ladder is at a distance of $10\sqrt{2}$ feet from the wall when it and the top are moving at the same rate.

EXAMPLES

1. Prove that if a particle moves so that the space described is proportional to the square of the time of description, the velocity will be proportional to the time and the rate of increase of the velocity will be constant.

2. A point moves in a fixed straight path so that $s = \sqrt{t}$; show that the acceleration is negative and proportional to the cube of the velocity.

3. If the path of a moving point is the sine curve

$$x = at, \quad y = b \sin at,$$

show (i) that the x-component of the velocity is constant; and (ii) that the acceleration of the point at any instant varies as the distance from the axis of x.

4. Show that the volume of a spherical soap bubble increases $4\pi r^2$ times as fast as the radius.

5. Sand is being poured on the ground from the orifice of an elevated pipe and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If sand is falling at the rate of a cubic cm. per second, how fast is the height of the pile increasing when the height is b cm. ?

6. A point source of light is hung a feet directly above a straight horizontal path on which a boy b feet in height is walking. How fast is the boy's shadow lengthening when he is walking away from the light at the rate of c feet per minute ?

7. A rod AB, 10 feet long, moves with its ends A and B on two perpendicular lines OX and OY respectively. If A is 8 feet from O and is moving away at the rate of 2 feet per second, find at what rate the end B is moving.

8. An inverted cone has a depth of 10 cm. and a base of radius 5 cm. Water is poured into it at the rate of $1\frac{1}{2}$ c.c. per minute. Find the rate at which the level of the water in the cone is rising when the depth is 4 cm.

9. A particle describes an ellipse whose semi-axes are 4 metres and 3 metres with a constant speed of 1 metre per second. Find the velocity of the foot of the perpendicular from the particle on the major axis, when the particle is at a distance of 1 metre from the major axis.

10. If force is equal to the space rate of change of kinetic energy and kinetic energy $= \frac{1}{2}mv^2$ prove that force $= ma$.

11. Prove that as x increases,

$$(a \sin x + b \cos x) / (c \sin x + d \cos x),$$

where a, b, c, d are constants, either increases for all values of x, or decreases for all value of x.

12. Find the range of values of x for which the function

$$x^3 - 6x^2 - 36x + 7$$

increases with x.

ANSWERS

5. $a / \pi b^2$ cm. per sec.
6. $bc / (a - b)$ ft. per min.
7. $2\frac{2}{3}$ ft. per sec.
8. $3/8\pi$ cm. per sec.
9. $\sqrt{2/11}$ m. per sec.
12. $x < -2$ and also $x > 6$.

4.4. Approximate Calculations. Small Increments.

Since $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx},$$

we must have, for small values of δx

$$\frac{\delta y}{\delta x} = \frac{dy}{dx} \quad \text{approximately;}$$

i.e. $$\delta y = \left(\frac{dy}{dx}\right) \delta x \quad \textit{approximately.}$$

This formula is very useful in calculating small changes, and is of immense importance in the theory of small errors, in physics, and in several other branches of science. It will be shown later that we can safely employ this formula whenever δx is so small that its second and higher powers can be neglected.

NOTE. If δx is an error in x, δx is called the **absolute error.** The **relative error** in x is $\delta x / x$. The **percentage error** is $100(\delta x / x)$. Thus if an error of 0.01 cm. is committed in measuring a length of 5 cm., the absolute error is 0.01 cm., the relative error is 0.002 and the percentage error is 0.2.

Ex. 1. If the radius of a spherical balloon increases by 0.1 percent, find approximately the percentage increase in the volume.

The volume = V, say, $= \frac{4}{3}\pi r^3$, where r is the radius.

By logarithmic differentiation,

$$\frac{1}{V}\frac{dV}{dr} = \frac{3}{r}.$$

Hence $\frac{dV}{V} = \frac{3}{r}\delta r$ approximately.

Multiplying both sides by 100 and putting 0.1 for 100 $\delta r/r$, we have at once :

the percentage increase in $V = 3 \times 0.1$ approximately.

Ex. 2. Given $\log_e 4 = 1.3863$, find $\log_e 4.01$.

$$\delta(\log x) = (1/x)\,\delta x = \tfrac{1}{4} \times 0.01 = 0.0025.$$

Therefore $\log_e 4.01 = 1.3863 + 0.0025 = 1.3888.$

EXAMPLES

1. Given $\log_{10} 4 = 0.6021$, calculate approximately $\log_{10} 4.04$, it being given that $\log_{10} x = 0.4343 \log_e x$.

2. Given that $\log_{10} e = 0.4343$, find $\log_{10} 10.1$.

3. The angular elevation of the top of a vertical flagstaff at a point on the ground d metres from the foot of the flagstaff is θ. Find the error in the height h of the flagstaff due to an error $\delta\theta$ in the angular elevation.

4. A gas expands isothermally. If the volume v and the pressure p are connected by the relation pv = constant, find the increase in pressure when the volume of a quantity of gas under a pressure of 20 kg. per square cm. is altered from 1000 cubic cm. to 999 cubic cm.

5. A tower is at a distance of 500 feet, and its top is observed to be at an elevation of 30°. Calculate the total height of the tower, supposing the height of the observer to be 5 feet. If the angle of elevation was really 30° 12', what error has crept into the calculated height of the tower ? Given 12' = 0.0035 radian approximately.

6. The radius of a sphere is found by measurement to be 18.5 inches with a possible error of 0.1 inch. Find the consequent error possible in (i) the surface area (ii) the volume, as calculated from this measurement.

7. The pressure p and the volume v of a gas are connected by the relation $pv^{1.4}$ = constant. Find the percentage increase in the pressure corresponding to a diminution of 1/2 per cent in the volume.

8. The time T of a complete oscillation of a simple pendulum of length l is given by the equation $T = 2\pi\sqrt{l/g}$, where g is a constant. Find the approximate error in the calculated value of T corresponding to an error of 2 per cent in the value of l.

9. With the usual notation, if Δ be the area of a triangle, prove that the error in Δ resulting from a small error in the measurement of c is given by

$$\delta\Delta = \frac{1}{4}\Delta\{1/s + 1/(s-a) + 1/(s-b) - 1/(s-c)\}\delta c.$$

ANSWERS

1. 0.6064.
2. 1.0043.
3. $d \sec^2\theta\,\delta\theta$.
4. 0.02 kg. per sq. cm.
5. 294 ft.; 0.79 per cent.
6. (1) $1.85 \times 8\pi$ sq. inches;
   (2) $0.4 \times (18.5)^2\pi$ cubic inches.
7. 0.7 per cent.
8. 1 per cent.

4.5. Application to the Theory of Equations. If $f(x)$ is a rational integral algebraic function of x, and the equation $f(x) = 0$ has two roots equal to a, then $f(x)$ must be of the form $(x-a)^2 F(x)$.

By actual differentiation we see that the equation $f'(x) = 0$ must in this case be

$$2(x-a)F(x) + (x-a)^2 F'(x) = 0$$

so that a is also a root of $f'(x) = 0$.

It is easy to show similarly that *whenever a is a multiple root of $f(x) = 0$*, it will also be a root of $f'(x) = 0$. Thus, by finding the G.C.M. of $f(x)$ and $f'(x)$, we can find such roots.

Ex. The equation $x^4 - x^3 - x + 1 = 0$ has one double root. Find it and hence solve the equation.

Here $\frac{d}{dx}(x^4 - x^3 - x + 1) = 4x^3 - 3x^2 - 1$

The G.C.M. of $4x^3 - 3x^2 - 1$ and $x^4 - x^3 - x + 1$ is $x - 1$.

Hence $(x-1)^2$ is a factor of $x^4 - x^3 - x + 1$.

By division, we find the other factor to be $x^2 + x + 1$.

This equated to zero gives the roots as $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$

Hence the four roots of $x^4 - x^3 - x + 1 = 0$ are

$$1,\ 1,\ -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i,\ -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\,i.$$

4.51. Approximate Solution of Equations. Newton's Method.* If we have to get a better value of a root of the equation $f(x)=0$, when it is known that its approximate value is a, the proposition of § 4.4 is useful.

For, if $a+\delta x$ be the root, then $f(a+\delta x)$ must be zero.

Also, as a is an approximate value of the root, δx must be small. So by § 4.4,

$$\delta f(x) = f'(x)\,\delta x \qquad \text{nearly,}$$

i.e. $$f(a+\delta x) - f(a) = f'(a)\,\delta x, \text{ nearly.}$$

Hence, we have, since $f(a+\delta x)=0$,

$$\delta x = -f(a)/f'(a), \text{ approximately.}$$

This determines δx, which added to a gives a better value of the root. It is obvious that $f(x)$ need not be algebraical.

NOTE. Taking $a+\delta x$ as the new approximate value of the root, the above process can be repeated to find a still better value of the root, and so on.

Ex. Find that root of the equation

$$x^4 - 12x + 7 = 0$$

which is approximately equal to 2.

Here $\qquad f'(x) = 4x^3 - 12.$

So $\qquad f'(2) = 20.$

Also $\qquad f'(2) = -1.$

Therefore $\qquad \delta x = +\,1/20 = 0.05$

Hence the root is approximately equal to 2.05.

[Repetition of the process a few times would show that the root is 2.047275567 correct to 9 decimal places. The method, however, becomes very laborious after one or two steps.]

---

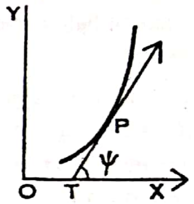
* Named after Isaac Newton (1642-1727), the great English mathematician, who was the first to invent the calculus (see the Historical Note at the end of this book). The method given here is the modification originally given by Joseph Raphson (1648-1715) a Fellow of the Royal Society, London. Hence this method is also called the "Newton-Raphson Method".

4.6. Applications to Geometry. Another meaning of the Differential Coefficient. The differential coefficient is useful in evaluating limits which cannot be found by mere substitution (Chap. XV), in finding maxima of functions (Chap. XIV) and in expanding functions of x in powers of x (Chap. VII).

In Geometry it is useful in finding tangents and normals of curves (Chaps. VIII and IX), the rate at which they bend (Chap. X) and their shapes (Chap. XI).

These various topics will be dealt with in due course. In particular, it will be shown that if $y=f(x)$ be a curve, then the tangent to it at any point (x, y) makes with the axis of x an angle y whose trigonometrical tangent is dy/dx, i.e.,

$$\tan\psi = \frac{dy}{dx}$$



This gives us a new meaning of the differential coefficient, which is of great importance in the application of differential calculus to Geometry.

EXAMPLES (MISCELLANEOUS)

1. If at time t the displacement s of a particle moving away from the origin is given by

$$s = a\sin t + b\cos t,$$

find the velocity and acceleration of the particle.

2. A stone is dropped into a calm lake, sending out a series of concentric ripples. If the radius of the outer ripple increases uniformly at the rate of c cm. per second, how rapidly is the disturbed area increasing t seconds after the stone hits the water ?

3. A military observer in an aeroplane is ascending at the rate of a miles an hour. If the radius of the earth be r miles, how fast is the

visible area of the earth's surface increasing in square miles per minute t minutes after the plane left the ground ?

4. Van der Waal's equation for a gas is

$$(p+av^{-2})(v-b)=k$$

where p is the pressure, v the volume, and a, b and k are constants. What is the change in volume per unit increase in the pressure ?

5. A particle moves along the parabola $y^2=2px$ in such a way that its projection on the y-axis has a constant velocity. Show that its projection on the x-axis moves with constant acceleration.

6. A revolving light 5 km. from a straight shore line makes two complete revolutions per minute. Find the velocity along the shore of the beam of light at the instant when it makes an angle of 60° with the shore line.

7. If the height of a cone increases by a per cent, its semivertical angle remaining the same, what is the approximate percentage increase (i) in the total area and (ii) in the volume, assuming that a is small ?

8. The angle of incidence φ and of refraction ψ are connected by the relation $\sin\varphi=\mu\sin\mu\psi$, where μ is a constant. If φ should change from 60° to $59^\circ 50'$, what would be the corresponding change in the value of ψ supposing that its former value was 45^0 ?

9. The area of a triangle is calculated from the angles A and C and the side b. If a small error δA is made in measuring A, show that the percentage error in the area is approximately

$$100\delta A.\sin C/\{\sin A.\sin(A+C)\}$$

10. The rate of flow Q of water per second over a sharp- edged notch of length l, the height of the general level of the water above the bottom of the notch being h, is given by the formula

$$Q=c(l-h/5)h^{3/2}$$

where c is a constant.

Show that for a small error δh in the measurement of h, the error δQ in Q is

$$\tfrac{1}{2}c(l-h/5)h^{3/2}.\delta h.$$

11. If $f'(x)>0$, for $a\le x\le b$ show that $f(x)$ is an increasing function of x in this interval.

Two points A and B lie on a fixed straight line. They are on opposite sides of, and at equal distances from, a fixed point O on the line. P is any fixed point not on the line AOB. Show that the sum of the distances of P from the points A and B is increased if the distance AB is increased.

12. Solve the following equations by finding first their multiple roots

(i) $4x^4-27x^2-25x-6=0$

(ii) $x^4+3x^3-3x^2-7x+6=0$

(iii) $8x^4-20x^3+18x^2-7x+1=0$

13. What condition must a and b satisfy in order that the equation $x^3+x^2+ax+b=0$ may have a double root ?

14. One root of the equation

$$x^4-12x^2-12x-3=0$$

is approximately equal to 4. Find it correctly to two decimal places.

15. Show by means of a rough graph that the least positive root of the equation $\tan x=\frac{1}{2}x$ between π and $3\pi/2$. Find this least root correct to four significant figures.

16. Show by starting from the rough approximation $x=\pi$ as a root of the equation $\sin x=ax$, where a is small, that a much better approximation can be obtained as

$$x=(1-a+a^2)\pi.$$

17. Show that for $x_2>x_1$,

$$\tan^{-1}x_2-\tan^{-1}x_1<x_2-x_1$$ [*Allahabad* 1995]

18. The angle A of a triangle ABC is found by measurement to be 63 degrees, and the area is calculated by the formula $\frac{1}{2}bc\sin A$. Find the percentage error in the calculated value of the area, due to an error of 15 minutes in the measured value of A.

19. A circle is drawn with its centre on a given parabola and touching its axis. Show that if the point of contact recede with a constant velocity from the vertex of the parabola, the rate of increase of the area of the circle is also constant.

20. $ABCD$ is a rectangular protractor in which AB = 6 inches, $BC = 2$ inches and O is the midpoint of AB. An angle BOP is indicated by a mark P on the edge CD. If in setting off an angle θ degrees, a mark is made 1/100 th of an inch along the edge from the correct spot, show that the error in the angle is

$$9\sin^2\theta/10\pi \text{ degrees.}$$

ANSWERS

1. $a\cos t - 2b\sin t\,;\, -a\sin t - 4b\cos 2t$.

2. $2\pi c^2 t$ sq. ft. per sec.

3. $120\pi\, ar^3/(60r + at)^2$.

4. $-(v-b)v^3/(pv^3 - av + 2ab)$.

5. $\frac{80}{3}\pi$ km/min.

7. $2a, 3a$.

8. Decrease by 5′4″.

12. (*i*) $-\frac{1}{2}, -\frac{1}{2}, 3$ and -2.

    (*ii*) 1, 1, −3 and −2.

    (*iii*) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and 1.

13. $4(3a-1)(3b-a^2) = (a-9b)^2$.

14. 3.91.

15. 4.275.

17. $3e^{-t/2}(6\cos 3t - \sin 3t)$.

18. $(5\pi/36)\cot 63^\circ$.

5

Successive Differentiation

5.1. Definition and Notation. If y be a function of x, its differential coefficient dy/dx will be, in general, a function of x which can be differentiated. The differential coefficient of dy/dx is called *the second differential coefficient* of y. Similarly, the differential coefficient of the second differential coefficient is called *the third differential coefficient*, and so on. The successive differential coefficients of y are denoted by

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots,$$

the nth differential coefficient of y being $\dfrac{d^ny}{dx^n}$.

This method of differentiating a function successively is called ***successive differentiation.***

Alternative methods of writing the nth differential coefficient are

$$\left(\frac{d}{dx}\right)^n y,\ D^n y,\ y_n,\ \frac{d^ny}{dx^n},\ y^n.$$

In the last case, the first, second, third etc. differential coefficients would be written as y', y'', y''', etc.

The value of a differential coefficient at $x = a$ is usually indicated by adding a suffix; thus $(y_n)_{x=a}$ or $(y_n)_a$. If $y = f(x)$, the same thing can also be indicated by $f^n(a)$.

Ex. 1. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that

$$\frac{d^2y}{dx^2} = -\frac{a}{(1-x^2)^{3/2}}$$

We have $$x\sqrt{1-y^2} + y\sqrt{1-x^2} = a, \tag{1}$$

Let $x = \cos\alpha$ and $y = \cos\beta$, then (1) becomes

$$\cos\alpha\sqrt{1-\cos^2\beta}+\cos\beta\sqrt{1-\cos^2\alpha}=a$$

or $\cos\alpha\sin\beta+\cos\beta\sin\alpha=a$ i.e. $\sin(\alpha+\beta)=a$

or $\alpha+\beta=\sin^{-1}a$ so that $\cos^{-1}x+\cos^{-1}y=\sin^{-1}a$

Differentiating w.r.t. x, $\dfrac{-1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx}=0$

or
$$\frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \qquad (2)$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2}=-\frac{\sqrt{1-x^2}\cdot\dfrac{1}{2\sqrt{1-y^2}}(-2y)\dfrac{dy}{dx}-\sqrt{1-y^2}\cdot\dfrac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2}$$

$$=-\frac{-\dfrac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\cdot y\left(-\dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right)+\dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\cdot x}{1-x^2} \qquad \text{[by (2)]}$$

$$=-\frac{y\sqrt{1-x^2}+x\sqrt{1-y^2}}{(1-x^2)^{3/2}}$$

Hence
$$\frac{d^2y}{dx^2}=\frac{-a}{(1-x^2)^{3/2}} \qquad \text{[by (1)]}$$

Ex. 2. If $y=a\cos(\log x)+b\sin(\log x)$, show that

$$x^2y_2+xy_1+y=0.$$

[*Avadh* **08 (1st part),** ***Bundelkhand*** **06, 11, 12 (1st part)**]

We have
$$y=a\cos(\log x)+b\sin(\log x) \qquad (1)$$

Therefore
$$y_1=[-a\sin(\log x)]\cdot\frac{1}{x}+[b\cos(\log x)]\cdot\frac{1}{x}$$

or
$$xy_1=-a\sin(\log x)+b\cos(\log x)$$

Differentiating again w.r.t. x, we get ,

$$xy_2+y_1=[-a\cos(\log x)]\cdot\frac{1}{x}+[-b\sin(\log x)]\cdot\frac{1}{x}$$

$$=-\frac{1}{x}[a\cos(\log x)+b\sin(\log x)]=-\frac{y}{x} \qquad \text{[using (1)]}$$

Hence
$$x^2y_2+xy_1+y=0.$$

Ex. 3. If $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, find $\dfrac{d^2y}{dx^2}$ at $\theta=\pi$.

[***Delhi*** **2008**]

Here
$$x=a(\theta-\sin\theta)\Rightarrow\frac{dx}{d\theta}=a(1-\cos\theta)$$

and
$$y=a(1-\cos\theta)\Rightarrow\frac{dy}{d\theta}=a\sin\theta$$

$$\therefore\quad\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{a\sin\theta}{a(1-\cos\theta)}$$

or
$$\frac{dy}{dx}=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}=\cot\tfrac{1}{2}\theta.$$

$$\therefore\quad\frac{d^2y}{dx^2}=-\operatorname{cosec}^2\tfrac{1}{2}\theta\cdot\frac{1}{2}\cdot\frac{d\theta}{dx}=-\frac{1}{2}\operatorname{cosec}^2\tfrac{1}{2}\theta\cdot\frac{1}{a(1-\cos\theta)}$$

or
$$\frac{d^2y}{dx^2}=-\frac{1}{4a}\operatorname{cosec}^4\tfrac{1}{2}\theta=-\frac{1}{4a}\quad\text{at}\quad\theta=\pi.$$

Ex. 4. If
$$p^2=a^2\cos^2\theta+b^2\sin^2\theta, \qquad (1)$$

prove that
$$p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}. \qquad \text{[\textit{Jabalpur} 03, 06]}$$

Differentiating (1) w.r.t. θ, we get

$$2p\frac{dp}{d\theta}=-2a^2\sin\theta\cos\theta+2b^2\sin\theta\cos\theta$$

$2\cos\theta\sin\theta=\sin 2\theta$

or
$$2p\frac{dp}{d\theta}=(b^2-a^2)\sin 2\theta \qquad (2)$$

Again differentiating (2) w.r.t. θ, we get

$$2p\frac{d^2p}{d\theta^2}+2\left(\frac{dp}{d\theta}\right)^2=2(b^2-a^2)\cos 2\theta$$

or
$$p\frac{d^2p}{d\theta^2}+\left(\frac{dp}{d\theta}\right)^2=(b^2-a^2)\cos 2\theta \qquad (3)$$

Multiplying (3) by p^2, we get

$$p^3\frac{d^2p}{d\theta^2}+p^2\left(\frac{dp}{d\theta}\right)^2=p^2(b^2-a^2)\cos 2\theta$$

i.e.
$$p^3\frac{d^2p}{d\theta^2}=p^2(b^2-a^2)\underline{\cos 2\theta}-\left(p\frac{dp}{d\theta}\right)^2$$

$$= p^2(b^2-a^2)(\cos^2\theta-\sin^2\theta)-\frac{(b^2-a^2)^2}{4}\sin^2 2\theta, \qquad \text{[Using (2)]}$$

$$= p^2(b^2\cos^2\theta+a^2\sin^2\theta)-p^2(a^2\cos^2\theta+b^2\sin^2\theta)$$
$$-(a^4+b^4-2a^2b^2)\sin^2\theta\cos^2\theta$$

$$=(a^2\cos^2\theta+b^2\sin^2\theta)(b^2\cos^2\theta+a^2\sin^2\theta)-p^4$$
$$-(a^4+b^4-2a^2b^2)\sin^2\theta\cos^2\theta, \qquad \text{[Using (1)]}$$

or $$p^4+p^3\frac{d^2p}{d\theta^2}=a^2b^2(\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta)$$

$$=a^2b^2(\cos^2\theta+\sin^2\theta)^2=a^2b^2$$

or $$p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}.$$ Hence Proved

EXAMPLES

1. If $y=A\sin mx+B\cos mx$, prove that $y_2+m^2y=0$.
2. If $y=e^{ax}\sin bx$, prove that $y_2-2ay_1+(a^2+b^2)y=0$. [*Kanpur* 2005]
3. If $y=Ae^{-kt}\cos(pt+c)$, show that
$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+n^2y=0 \quad \text{where} \quad n^2=p^2+k^2.$$
[*Lucknow* 11, *Kashi Vidyapeeth* 12]
4. If $y=\sin(\sin x)$, show that $y_2+y_1\tan x+y\cos^2 x=0$. [*Patna* 2005]
5. If $y=a\sin(\log x)$, show that $x^2y_2+xy_1+y=0$.
6. If $y^3-3ax^2+x^3=0$, show that $\frac{d^2y}{dx^2}+\frac{2a^2x^2}{y^5}=0$. [*Agra* 03]
7. If $y=\frac{\log x}{x}$, show that $y_2=\frac{3-2\log x}{x^3}$. [*Agra* 2008]
8. If $ay=\sin(x+y)$, prove that $y[1+(dy/dx)]^3=0$.
9. If $x=\log\phi$, $y=\phi^2-1$, find $\frac{d^2y}{dx^2}$.
10. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, show that

$D^2y=\Delta(hx+by+f)^{-3}$ where $\Delta=abc+2fgh-af^2-bg^2-ch^2$.

ANSWERS

9. $4\phi^2$.

5.2. Standard Results. Calculation of the *n*th derivative.

(1) If $y=(ax+b)^m$, then $y_1=m\,.\,a(ax+b)^{m-1}$,

$y_2=m(m-1)\,.\,a^2(ax+b)^{m-2}$, etc. Hence

$$D^n(ax+b)^m=m(m-1)(m-2)\ldots(m-n+1)\,a^n(ax+b)^{m-n}$$

In particular,

$$D^n(ax+b)^{-1}=(-1)^n\,n!\,a^n(ax+b)^{-n-1}.$$

If m is a positive integer, the $(m+1)$th and all the successive differential coefficients of $(ax+b)^m$ would be zero.

(2) If $y=e^{ax}$, then $y_1=ae^{ax}$, $y_2=a^2e^{ax}$, etc.

In general, $D^ne^{ax}=a^ne^{ax}$.

(3) Similarly $D^n a^x=(\log a)^n a^x$.

(4) If $y=\log(ax+b)$, then $y_1=a(ax+b)^{-1}$,

$y_2=(-1)\,a^2(ax+b)^{-2}$, $y_3=(-1)\,a^3(ax+b)^{-3}$, etc.

In general, $$D^n\log(ax+b)=\frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}.$$

(5) If $y=\sin(ax+b)$,

$$y_1=a\cos(ax+b)=a\sin(ax+b+\tfrac{1}{2}\pi),$$
$$y_2=a^2\cos(ax+b+\tfrac{1}{2}\pi)=a^2\sin(ax+b+\pi),$$
$$y_3=a^3\sin(ax+b+\tfrac{3}{2}\pi); \qquad \text{etc.}$$

In general,

$$D^n\sin(ax+b)=a^n\sin(ax+b+\tfrac{1}{2}n\pi).$$

(6) Similarly

$$D^n\cos(ax+b)=a^n\cos(ax+b+\tfrac{1}{2}n\pi).$$

Corollaries. Putting $a=1$ and $b=0$, we have

$$D^n\sin x=\sin(x+\tfrac{1}{2}n\pi),$$

and $$D^n\cos x=\cos(x+\tfrac{1}{2}n\pi).$$

(7) If $y = e^{ax} \sin(bx + c)$, then

$$y_1 = e^{ax} b \cos(bx + c) + a e^{ax} \sin(bx + c).$$

Putting $a = r \cos\phi$ and $b = r \sin\phi$, we have

$$y_1 = r e^{ax} \sin(bx + c + \phi).$$

Similarly $y_2 = r^2 e^{ax} \sin(bx + c + 2\phi)$; etc. In general,

$$\boldsymbol{D^n \{e^{ax} \sin(bx + c)\} = r^n e^{ax} \sin(bx + c + n\phi)},$$

where $\boldsymbol{r = (a^2 + b^2)^{1/2}}$, **and** $\boldsymbol{\phi = \tan^{-1}(b/a)}$.

(8) Similarly

$$\boldsymbol{D^n \{e^{ax} \cos(bx + c)\} = r^n e^{ax} \cos(bx + c + n\phi)},$$

where r and ϕ have the same meaning as before.

5.3. Decomposition into a Sum. Before applying the above standard results to particular functions, it is often necessary to break up the given function into a sum of suitable functions. Some methods for this are dealt with below.

(1) Partial Fractions. For finding the nth differential coefficient of a fraction whose numerator and denominator are both rational integral algebraic functions, the given fraction must be decomposed into its partial fractions.

Ex. Find the nth derivative of $\dfrac{x^3}{(x^2 - 3x + 2)}$. [*Bilaspur* 2009]

We have $$\frac{x^3}{(x^2 - 3x + 2)} = \frac{x^3}{(x-1)(x-2)} = x + 3 - \frac{1}{x-1} + \frac{8}{x-2}$$

by breaking into partial fractions.

Therefore, if $n > 1$, $$\frac{d^n}{dx^n} \frac{x^3}{(x-1)(x-2)}$$
$$= (-1)^{n+1} (n!) \left\{ \frac{1}{(x-1)^{n+1}} - \frac{8}{(x-2)^{n+1}} \right\}.$$

(2) Use of De Moivre's Theorem. Even when we cannot break up the denominator of a given algebraic fraction into real linear factors, the above method can be used

after resolving the denominator into its linear factors, real or imaginary.

Ex. 1. Find the nth differential coefficient of $\tan^{-1} x$.

[*Delhi* 07, *Lucknow* 10]

If $y = \tan^{-1} x$, then $y_1 = 1/(1 + x^2)$.

Now $$\frac{1}{1+x^2} = \frac{1}{(x+i)(x-i)} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

Therefore $$y_n = D^{n-1}\left(\frac{1}{1+x^2}\right) = \frac{(-1)^{n-1}(n-1)!}{2i}\left\{\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right\}$$

To get rid of the 'i' in the above derivative, put
$x = r\cos\phi, 1 = r \sin\phi$; then

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left\{\frac{1}{(\cos\phi - i\sin\phi)^n} - \frac{1}{(\cos\phi + i\sin\phi)^n}\right\}$$
$$= (-1)^{n-1}(n-1)!(1/2i) r^{-n} \{(\cos\phi - i \sin\phi)^{-n} - (\cos\phi + i\sin\phi)^{-n}\}$$
$$= (-1)^{n-1}(n-1)!(1/2i) r^{-n} \{(\cos n\phi + i \sin n\phi) - (\cos n\phi - i \sin n\phi)\}$$
$$= (-1)^{n-1}(n-1)! r^{-n} \sin n\phi$$

But $r^{-n} = \sin^n \phi$ [since $1 = r \sin\phi$]

Hence $D^n \tan^{-1} x = (-1)^{n-1}(n-1)! \sin^n \phi \sin n\phi$,

where $\phi = \tan^{-1} x$.

Ex. 2. Find the nth differential coefficient of $\sin^5 x \cos^3 x$.

Let $\cos x + i \sin x = z$; then $\cos x - i \sin x = z^{-1}$.

Therefore $2\cos x = z + z^{-1}, 2i\sin x = z - z^{-1}$.

Also, by De Moivre's Theorem,

$$2\cos px = z^p + z^{-p}; \qquad 2i\sin px = z^p - z^{-p}.$$

Therefore $2^5 2^3 i^5 \sin^5 x \cos^3 x = (z - z^{-1})^5 (z + z^{-1})^3$
$$= (z^8 - z^{-8}) - 2(z^6 - z^{-6}) - 2(z^4 - z^{-4}) + 6(z^2 - z^{-2})$$
$$= 2i\sin 8x - 4i \sin 6x - 4i\sin 4x + 12 i \sin 2x.$$

Hence

$$D^n(\sin^5 x \cos^3 x) = 2^{-7}\{8^n \sin(8x + \tfrac{1}{2}n\pi) - 2.6^n \sin(6x + \tfrac{1}{2}n\pi)$$
$$- 2.4^n \sin(4x + \tfrac{1}{2}n\pi) + 6.2^n \sin(2x + \tfrac{1}{2}n\pi)\}.$$

(3) Trigonometrical transformation. It is possible to break up products of powers of sines and cosines into a sum by Trigonometry.

Ex. 1. Find the nth differential coefficient of

$$y=\cos x \cos 2x \cos 3x$$

Here $y=\cos x \cos 2x \cos 3x$

$$=\tfrac{1}{2}(2\cos 3x.\cos x).\cos 2x=\tfrac{1}{2}(\cos 4x+\cos 2x).\cos 2x$$

$$=\tfrac{1}{4}(2\cos 4x \cos 2x+2\cos^2 2x)$$

$$=\tfrac{1}{4}(\cos 6x+\cos 2x+1+\cos 4x)$$

i.e. $y=\tfrac{1}{4}(\cos 6x+\cos 4x+\cos 2x+1)$ so that

$$y_n=\frac{1}{4}\left[6^n\cos\left(6x+n\frac{\pi}{2}\right)+4^n\cos\left(4x+n\frac{\pi}{2}\right)+2^n\cos\left(2x+n\frac{\pi}{2}\right)\right]$$

Ex. 2. If $y=\sin mx+\cos mx$, show that

$y_n=m^n[1+(-1)^n\sin 2mx]^{1/2}$ **[*Patna* 06, *Bilaspur* 10, 13]**

Since $D^n\{\sin(ax+b)\}=a^n\sin(ax+b+\tfrac{1}{2}n\pi)$

and $D^n\{\cos(ax+b)\}=a^n\cos(ax+b+\tfrac{1}{2}n\pi)$,

we have $y_n=m^n\sin(mx+\tfrac{1}{2}n\pi)+m^n\cos(mx+\tfrac{1}{2}n\pi)$

$$=m^n[\{\sin(mx+\tfrac{1}{2}n\pi)+\cos(mx+\tfrac{1}{2}n\pi)\}^2]^{1/2}$$

$$=m^n[\sin^2(mx+\tfrac{1}{2}n\pi)+\cos^2(mx+\tfrac{1}{2}n\pi)$$

$$+2\sin(mx+\tfrac{1}{2}n\pi)\cos(mx+\tfrac{1}{2}n\pi)]^{1/2}$$

$$=m^n[1+\sin 2(mx+\tfrac{1}{2}n\pi)]^{1/2}=m^n[1+\sin(n\pi+2mx)]^{1/2}$$

$=m^n[1+(-1)^n\sin 2mx]^{1/2}$, since $\sin(n\pi+\theta)=(-1)^n\sin\theta$.

EXAMPLES

Find the nth differential coefficients of

1. $\sin^3 x$. 2. $\sin 3x\sin 2x$. **[*Lucknow* 04]**
3. $\cos^2 x\sin^3 x$. 4. $e^{ax}\cos^2 x\sin x$. **[*Lucknow* 12]**
5. $(ax+b)/(cx+d)$ **[*Lucknow* 06, *Kashi Vidyapeeth* 10]**
6. $\dfrac{1}{1-5x+6x^2}$. 7. $\tan^{-1}\{2x/(1-x^2)\}$. **[*Lucknow* 11]**

8. If $y=\tan^{-1}[(1+x)/(1-x)]$, find y_n. **[*Patna* 2011]**
9. If $y=e^{ax}\sin bx$, then show that $y_{n+1}=2ay_n-(a^2+b^2)y_{n-1}$.
[*Kumaon* 2013]
10. If $y=\dfrac{x}{x^2+a^2}$, prove that $y_n=\dfrac{(-1)^n n!}{a^{n+1}}\sin^{n+1}\theta\cdot\cos(n+1)\theta$,
where $\theta=\tan^{-1}(a/x)$. **[*Kanpur* 09, *Kumaon* 13]**
11. If $y=(x^2-1)^n$, prove that $y_{2n}(2n)!$.
[*Agra* 03, 04, 06, *Jabalpur* 04, *Rohilkhand* 06]

[HINT. $y=x^{2n}-{}^nC_1x^{2n-2}+{}^nC_2x^{2n-4}-\ldots.+(-1)^n$.]

ANSWERS

1. $\tfrac{3}{4}\sin(x+\tfrac{1}{2}n\pi)-(\tfrac{3}{4})^n\sin(3x+\tfrac{1}{2}n\pi)$.
2. $\tfrac{1}{2}\{\cos(x+\tfrac{1}{2}n\pi)-5^n\cos(5x+\tfrac{1}{2}n\pi)\}$.
3. $\tfrac{1}{16}\{2\sin(x+\tfrac{1}{2}n\pi)+3^n\sin(3x+\tfrac{1}{2}n\pi)-5^n\sin(5x+\tfrac{1}{2}n\pi)\}$.
4. $\tfrac{1}{4}(a^2+1)^{n/2}e^{ax}\sin\{x+n\tan^{-1}(1/a)\}+$
$\tfrac{1}{4}(a^2+9)^{n/2}e^{ax}\sin\{3x+n\tan^{-1}(3/a)\}$
5. $(-1)^n(n!)c^{n-1}(bc-ad)(cx+d)^{-n-1}$.
6. $(n!)\{3^{n+1}(1-3x)^{-n-1}-2^{n+1}(1-2x)^{-n-1}\}$
7. $y_n=2(-1)^{n-1}(n-1)!\sin^n\theta\cdot\sin n\theta$, where $\theta=\tan^{-1}(1/x)$.
8. $y_n=(-1)^n(n-1)!\sin^n\theta\cdot\sin n\theta$, where $\tan\theta=\tfrac{1}{x}$.

5.4. Leibnitz's Theorem. This theorem is useful ***for finding the nth differential coefficient of the product of two functions.*** It is as follows:

If u and v be any two functions of x, having differential coefficients of the nth order, then

$$D^n(uv)=(D^n u).v+{}^nC_1D^{n-1}u.Dv+{}^nC_2D^{n-2}u.D^2v$$
$$+\ldots+{}^nC_rD^{n-r}u.D^rv+\ldots+u.D^nv.$$

[*Rewa* 04, *Delhi* 04, 05, *Avadh* 05, *Agra* 07, *Kumaon* 08, *Patna* 05, 07, 09, *Meerut* 08, 09, *Magadh* 05, 07, 11, *Allahabad* 11]

We shall prove this theorem by mathematical induction. Assume that the above result is true for a particular value of n. Then, differentiating with respect to x, we have

$$D^{n+1}(uv)=\{(D^{n+1}u).v+D^n u.Dv\}$$
$$+{}^nC_1\{D^n u.Dv+D^{n-1}u D^2 v\}+\ldots$$
$$+{}^nC_r\{D^{n-r+1}u.D^r v+D^{n-r}u.D^{r+1}v\}+\ldots$$
$$+\{Du.D^n v+u.D^{n+1}v\}.$$

Rearranging, we have

$$D^{n+1}(uv)=(D^{n+1}u).v+(1+{}^nC_1)(D^n u.Dv)+\ldots$$
$$+({}^nC_r+{}^nC_{r+1})(D^{n-r}u.D^{r+1}v)+\ldots+u.D^{n+1}v.$$

But $\quad {}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}.$

Hence $D^{n+1}(uv)=(D^{n+1}u).v+{}^{n+1}C_1\,D^n u.Dv+\ldots$
$$+{}^{n+1}C_{r+1}\,D^{n-r}u.D^{r+1}v+\ldots+u.D^{n+1}v.$$

Therefore, if the theorem is true for any value of n, it is true for the next value of n.

But it is easy to see that it holds for $n=2$, for

$$D(uv)=(Du).v+u.Dv$$

Therefore $\quad D^2(uv)=(D^2u).v+2Du.Dv+u.D^2v$

Hence, by induction, the theorem must be true when $n=3$, and also when $n=4$; and so on. Thus it must be true for every positive integral value of n.

Ex. 1. If $\quad y=e^{a\sin^{-1}x}$ (or $x=\sin[(\log y)/a]$), prove that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$$

[*Jabalpur* 03, *Indore* 03, *Guwahati* 05, *Allahabad* 05, *Avadh* 06, *Agra* 06, 08, *Rohilkhand* 03, 10 (1st part), *Raipur* 15]

Since $\quad y=e^{a\sin^{-1}x}$, we have

$$y_1=e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}} \qquad \text{or} \qquad (1-x^2)y_1^{\,2}=a^2y^2.$$

Differentiating this w.r.t. x, we get

$$(1-x^2)2y_1y_2-2xy_1^{\,2}=2a^2yy_1$$

or $\quad (1-x^2)y_2-xy_1=a^2y \qquad$ [Dividing by $2y_1$]

Applying Leibnitz 's theorem, we get

$$(1-x^2)y_{n+2}-(2x)n\,y_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n=a^2y_n$$

or $\quad (1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$

Ex. 2. If $\quad y^{1/m}+y^{-1/m}=2x$, prove that

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$$

[*Meerut* 04, *Rewa* 05, *Allahabad* 06, *Indore* 04, 07, *Agra* 09, *Lucknow* 04, 05, 08, 10, *Kashi Vidyapeeth* 10, *Rohilkhand* 06, 11]

We have $\quad y^{1/m}+y^{-1/m}=2x \quad$ i.e. $\quad y^{2/m}-2xy^{1/m}+1=0.$

Solving this as a quadratic equation in $y^{1/m}$, we have

$$y^{1/m}=\frac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2-1} \quad \text{i.e.} \quad y=(x\pm\sqrt{x^2-1})^m$$

Therefore $\quad y_1=m(x\pm\sqrt{x^2-1})^{m-1}\left(1\pm\frac{x}{\sqrt{x^2-1}}\right)$

or $\quad y_1=\dfrac{m(x\pm\sqrt{x^2-1})^m}{\sqrt{x^2-1}}$

Squaring both sides, we get $\quad (x^2-1)y_1^{\,2}=m^2y^2.$

Differentiating again w.r.t. x, we get

$$2(x^2-1)y_1y_2+2xy_1^{\,2}=2m^2yy_1$$

or $\quad (x^2-1)y_2+xy_1-m^2y=0$

Applying Leibnitz's Theorem, we get

$$(x^2-1)y_{n+2}+{}^nC_1y_{n+1}.2x+{}^nC_2y_n.2+xy_{n+1}+{}^nC_1y_n-m^2y_n=0$$

or $\quad (x^2-1)y_{n+2}+(2x)ny_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n-m^2y_n=0$

or $\quad (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$

Ex. 3. Differentiate n times the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+a^2y=0 \text{ with respect to } x.$$

[*Lucknow* 2009 (1st part)]

Here $D^n\{(1-x^2)y_2\}$

$$=(1-x^2)y_{n+2}+n.(-2x)y_{n+1}+\{n(n-1)/2!\}(-2)y_n$$

$D^n(-xy_1)=-xy_{n+1}-ny_n,$

$D^n(a^2y)=a^2y_n.$

Adding $\quad 0=(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-a^2)y_n$

i.e. $\quad (1-x^2)\dfrac{d^{n+2}y}{dx^{n+2}}-(2n+1)x\dfrac{d^{n+1}y}{dx^{n+1}}-(n^2-a^2)\dfrac{d^ny}{dx^n}=0.$

Ex. 4. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, prove that $I_n = n I_{n-1} + (n-1)!$.

[***Gorakhpur*** **06,** ***Rohilkhand*** **08,** ***Avadh*** **11,** ***Lucknow*** **12]**

Hence prove that

$$I_n = n!\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+ \ldots\ldots + \frac{1}{2}+1+\log x\right]$$

[***Agra*** **06,** ***Purvanchal*** **08,** ***Magadh*** **08,** ***Kashi Vidyapeeth*** **08,** ***Kumaon*** **09]**

$$I_n = \frac{d^n}{dx^n}(x^n \log x) = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}[x^{n-1} + n x^{n-1} \log x]$$

$$= \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}) + n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1} \log x)$$

$$= n I_{n-1} + (n-1)!.$$ Hence Proved

We have $$I_n = n I_{n-1} + (n-1)! \qquad (1)$$

Replacing n by $n-1$ in (1), we get

$$I_{n-1} = (n-1) I_{n-2} + (n-2)! \qquad (2)$$

Substituting (2) in (1), we have

$$I_n = n[(n-1) I_{n-2} + (n-2)!] + (n-1)!$$

$$= n(n-1) I_{n-2} + n(n-2)! + (n-1)!$$

$$= n(n-1) I_{n-2} + \frac{n(n-1)(n-2)!}{n-1} + \frac{n(n-1)!}{n}$$

$$= n(n-1) I_{n-2} + \frac{n!}{n-1} + \frac{n!}{n} = n!\left[\frac{1}{n}+\frac{1}{n-1}+\frac{n(n-1)}{n!} I_{n-2}\right]$$

$$= n!\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{(n-2)!} I_{n-2}\right]$$

Repeating this process, we have

$$I_n = n!\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+ \ldots\ldots + \frac{1}{3}+\frac{1}{2}+\frac{I_1}{1!}\right]$$

$$= n!\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+ \ldots\ldots + \frac{1}{3}+\frac{1}{2}+1+\log x\right]$$

which is the desired result.

EXAMPLES

Find the nth differential coefficients of

1. $e^x (ax+b)^2$. 2. $\sin x \log (ax+b)$.

3. $x^3 \cos x$. [***Meerut*** **2010]**

4. Find the fourth derivative of $e^x \log x$. [***Lucknow*** **2008]**

5. If $y = x^2 e^x$, show that

$$\frac{d^n y}{dx^n} = \frac{1}{2} n(n-1)\frac{d^2 y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)\, y.$$

[***Bundelkhand*** **08,** ***Lucknow*** **12]**

6. If $y = x^{n-1} \log x$, prove that $y_n = (n-1)!/x$.

[***Sagar*** **03,** ***Patna*** **04,** ***Bilaspur*** **08,** ***Agra*** **10]**

7. If $y = (x^2-1)^n$, prove that

$$(x^2-1)\, y_{n+2} + 2x\, y_{n+1} - n(n+1)\, y_n = 0.$$

[***Rewa*** **04,** ***Rohilkhand*** **06,** ***Patna*** **06,** ***Meerut*** **08,** ***Magadh*** **10, 11]**

8. If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0.$$

[***Lucknow*** **11,** ***Kashi Vidyapeeth*** **12]**

9. If $\cos^{-1}(y/b) = \log (x/n)^n$, prove that

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0.$$

[***Agra*** **04,** ***Indore*** **06,** ***Lucknow*** **06, 07,** ***Raipur*** **10]**

10. If $y = e^{\tan^{-1} x}$, prove that

$$(1+x^2)\, y_{n+2} + \{2(n+1)x - 1\}\, y_{n+1} + n(n+1)\, y_n = 0.$$

[***Delhi*** **05,** ***Avadh*** **10,** ***Allahabad*** **12]**

11. If $y = \tan^{-1} x$, prove that [***Avadh*** **05,** ***Allahabad*** **09]**

$$(1+x^2)\, y_{n+2} + 2(n+1)\, x\, y_{n+1} + n(n+1)\, y_n = 0.$$

12. If $y = \sin^{-1} x$, prove that

$$(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0.$$

[***Rewa*** **03,** ***Lucknow*** **05,** ***Bundelkhand*** **11]**

13. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that

$$(1-x^2)\, y_{n+1} - (2n+1)\, x\, y_n - n^2 y_n = 0.$$

[***Allahabad*** **04 (1st part),** ***Kanpur*** **10]**

14. If $y=\sin x\log(x^2+2x+1)$, prove that
$$(x+1)^2 y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0.$$
[*Kumaon* 09, 12]

15. If $x+y=1$, prove that
$$\frac{d^n}{dx^n}(x^n y^n)=n!\{y^n-({}^nC_1)^2 y^{n-1}x+({}^nC_2)^2 y^{n-2}x^2+\ldots+(-1)^n x^n\}.$$
[*Rohilkhand* 07, *Kanpur* 08, *Patna* 08]

16. By forming in two different ways the *n*th derivative of x^{2n}, prove that
$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}+\ldots=\frac{(2n!)}{(n!)^2}.$$
[HINT: First find the *n*th derivative of the product of x^n and x^n; next that of x^{2n}.]
[*Delhi* 03, *Jabalpur* 06]

ANSWERS

1. $e^x\{(ax+b)^3+3an(ax+b)^2+3n(n-1)a^2(ax+b)$
$+n(n-1)(n-2)a^3\}$.

2. $(-1)^{n-1}\{(n-1)!a^n(ax+b)^{-n}\sin x-{}^nC_1(n-2)!a^{n-1}(ax+b)^{-n+1}\cos x$
$-{}^nC_2(n-3)!a^{n-2}(ax+b)^{-n+2}\sin x$
$+{}^nC_r(-1)^r(n-r-1)!\,a^{n-r}(ax+b)^{-n+r}\sin(x+\tfrac{1}{2}r\pi)+\ldots\}$
$+\log(ax+b)\sin(x+\tfrac{1}{2}n\pi)$.

3. $x\{x^2-3n(n-1)\}\cos(x+\tfrac{1}{2}n\pi)+n\{3x^2-(n-1)(n-2)\sin(x+\tfrac{1}{2}n\pi)\}$

4. $e^x(\log x+4x^{-1}-6x^{-2}+8x^{-3}-6x^{-4})$.

5.5. Value of the *n*th differential coefficient for special values of *x*. Sometimes, although we may not be able to find the *n*th differential coefficient in a compact form for the general value of *x*, we can find it for some special value of *x*. The procedure will be clear from the following examples.

Ex. 1. If $y=\sin(m\sin^{-1}x)$, show that
$$(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-(n^2-m^2)y_n=0.$$
[*Jabalpur* 04, *Indore* 05, *Delhi* 06, *Avadh* 03, 07, *Magadh* 09, *Kashi Vidyapeeth* 11, *Raipur* 14]

and also find $y_n(0)$.

[*Gorakhpur* 03, *Poorvanchal* 04, *Himachal* 04, 06, 07, *Agra* 11]

We have $y=\sin(m\sin^{-1}x)$ so that
$$y_1=\cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}},\tag{1}$$
or
$$\sqrt{1-x^2}\,.\,y_1=m\cos(m\sin^{-1}x).$$
Differentiating again w.r.t. *x*, we get
$$\sqrt{1-x^2}\cdot y_2-\frac{x}{\sqrt{1-x^2}}\,y_1=m(-1)\sin(m\sin^{-1}x)\times\frac{m}{\sqrt{1-x^2}}$$
or
$$(1-x^2)y_2-x\,y_1+m^2y=0\tag{2}$$
Differentiating this *n* times by Leibnitz's Theorem, we have
$$(1-x^2)y_{n+2}+{}^nC_1(-2x)y_{n+1}+{}^nC_2(-2)y_n-x\,y_{n+1}-1.\,{}^nC_1\,y_n+m^2y_n=0.$$
or $(1-x^2)y_{n+2}-2nx\,y_{n+1}-n(n-1)y_n-x\,y_{n+1}-n\,y_n+m^2y_n=0.$

or
$$(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-(n^2-m^2)y_n=0\tag{3}$$
Putting $x=0$ in (3), this gives
$$y_{n+2}(0)=(n^2-m^2)y_n(0)\tag{4}$$
But by (1) and (2), $y_1(0)=m$ and $y_2(0)=0$.

Hence, by (4) y_2, y_4, y_6 ...... are all zero at $x=0$.

But if *n* is odd,
$$y_n(0)=\{(n-2)^2-m^2\}\,y_{n-2}(0)$$
$$=\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}y_{n-4}(0)=\text{etc.}$$
So $y_n(0)=\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\{(n-6)^2-m^2\}\ldots\{1^2-m^2\}\,m.$

Ex. 2. If $y=[x+\sqrt{1+x^2}]^m$, find $y_n(0)$.

[*Agra* 04, *Delhi* 04 (2nd part), *Jabalpur* 05, *Meerut* 06, 07, 09, *Kumaon* 10, *Avadh* 04, 09, 11, *Raipur* 12 (2nd part), *Bundelkhand* 14, (2nd part)]

Here
$$y=[x+\sqrt{1+x^2}]^m\tag{1}$$
Then
$$y_1=m[x+\sqrt{1+x^2}]^{m-1}\left[1+\frac{x}{\sqrt{1+x^2}}\right]=m\frac{[x+\sqrt{1+x^2}]^m}{\sqrt{1+x^2}}$$
or
$$\sqrt{1+x^2}\;y_1=m\,y\tag{2}$$
Differentiating w.r.t. *x*, we get $\sqrt{1+x^2}\;y_2+\dfrac{x}{\sqrt{1+x^2}}\cdot y_1=m\,y_1$

or $$(1+x^2)\,y_2+x\,y_1=m\sqrt{1+x^2}\cdot y_1=m\cdot m\,y \quad \text{[using (2)]}$$

or $$(1+x^2)\,y_2+x\,y_1-m^2\,y=0 \tag{3}$$

Differentiating this n times using Leibnitz's Theorem, we get

$$(1+x^2)\,y_{n+2}+2\,n\,x\,y_{n+1}+n\,(n-1)\,y_n+x\,y_{n+1}+n\,y_n-m^2\,y_n=0$$

or $$(1+x^2)\,y_{n+2}+(2\,n+1)\,x\,y_{n+1}+(n^2-m^2)\,y_n=0 \tag{4}$$

Putting $x=0$ in (1), (2), (3) and (4), we get

$$y\,(0)=1,\ y_1\,(0)=m,\ y_2\,(0)=m^2 \qquad \text{and}$$

$$y_{n+2}=(m^2-n^2)\,y_n\,(0) \tag{5}$$

Putting $n=1, 2, 3, 4$ etc. in (5), we get

$$y_3(0)=(m^2-1^2)\,y_1(0)=m\,(m^2-1^2)$$

$$y_4(0)=(m^2-2^2)\,y_2(0)=m^2(m^2-2^2)$$

$$y_5(0)=(m^2-3^2)\,y_3(0)=m\,(m^2-1^2)(m^2-3^2)$$

$$y_6(0)=(m^2-4^2)\,y_4(0)=m^2(m^2-2^2)(m^2-4^2)$$

In general, when n is odd, then

$$y_n(0)=[m^2-(n-2)^2][m^2-(n-4)^2]\,..........\,(m^2-1^2)\,m$$

and when n is even, then

$$y_n(0)=[m^2-(n-2)^2][m^2-(n-4)^2]\,..........\,(m^2-2^2)\,m^2.$$

EXAMPLES

1. If $y=[\log\{x+\sqrt{1+x^2}\}]^2$, find $(y_n)_0$.

[*Rohilkhand* 04, *Meerut* 05, *Kashi Vidyapeeth* 09 (2nd part)]

2. If $y=\tan^{-1}x$, then prove that $(1+x^2)\,y_2+2\,x\,y_1=0$.

Find the value of $y_n(0)$ and show that when $n=2p,\ 4p+1, 4p+3$, then its value is $0, (n-1)!, -(n-1)!$ respectively.

[*Agra* 02, *Jabalpur* 07]

3. If $y=\frac{1}{2}(\tan^{-1}x)^2$, show that

$$(y_{n+2})_0+2\,n^2(y_n)_0+n\,(n-1)^2\,(n-2)(y_{n-2})_0=0.$$

4. If $y=(\sin^{-1}x)^2$, show that

$$(1-x^2)\,y_{n+2}-(2r.+1)\,x\,y_{n+1}-n^2\,y_n=0.$$

[*Delhi* 05, *Agra* 08, *Allahabad* 12]

Also find $y_n(0)$. **[*Delhi* 2005]**

5. If $y=\cos\,(m\sin^{-1}x)$, find $y_n(0)$. **[*Himachal* 2005]**

6. If $y=e^{a\sin^{-1}x}$, prove that

$$(1-x^2)y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)y_n=0$$

and hence find the value of y_n at $x=0$.

[*Patna* 04, *Gorakhpur* 05, *Agra* 06, 08, *Rohilkhand* 10, 12]

ANSWERS,

1. 0 when n is odd, and

$(-1)^{(n-2)/2}(n-2)^2(n-4)^2(n-6)^2....2^2\cdot 2$ when n is even.

4. 0 when n is odd; $(n-2)^2(n-4)^2....4^2.2^2.2$ when n is even.

5. 0 when n is odd ;

$m^2\,(2^2-m^2)(4^2-m^2)....\{(n-2)^2-m^2\}$ when n is even.

6. $[(n-2)^2+a^2][(n-4)^2+a^2].....(3^2+a^2)(1+a^2).a$ when n is odd.

$[(n-2)^2+a^2][(n-4)^2+a^2]....(4^2+a^2)(2+a^2).a^2$ when n is even

6

Convergence of Series

6.1. Sequences. We have seen in Chapter I that a sequence $s_1, s_2, \dots, s_n, \dots$, has a limit s if $|s_n - s|$ can be made as small as we please by taking n sufficiently large. A sequence for which s has a definite finite limit is called a *convergent* sequence. Keeping in mind the definition of limit (§ 2.5), we can give the following definition.

A sequence $s_1, s_2, \dots, s_n, \dots$, is said to be* CONVERGENT *if there exists a number s with the property that for any given positive number ε, however small, we can find a number N such that

$$|s_n - s| < \varepsilon, \text{ for all } n > N.$$

We say that $\{s_n\}$ converges to s. The number s is called the *limit* of the sequence $\{s_n\}$. A sequence for which such a limit does not exist, is called a ***non-convergent (or divergent)*** sequence.

Ex. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ converges to the limit 1. The sequence $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots$ converges to 0. The sequences $1, 2^2, 3^3, 4^2, \dots$, and $1, -1, 1, -1, \dots$ are both non-convergent.

6.11. Cauchy Sequence. Let $\{s_n\} = s_1, s_2, \dots, s_n, \dots$ be a given sequence. If for every positive number ε we can find a number N such that

$$|s_m - s_n| < \varepsilon \text{ for all } m, n > N.$$

then the sequence $\{s_n\}$ is called a Caucny sequence.

To put it simply, the terms of a Cauchy sequence become *arbitrarily close to each other* as the sequence progresses i.e. if we drop a finite number of terms from the beginning of the sequence, we can make the distance between any two of the remaining terms arbitrarily small.

It is easy to show that ***every convergent sequence is a Cauchy sequence***. Let $\{s_n\}$ be a sequence which converges to s. Choose a positive number ε. Then we can find a number N such that

$$|s_n - s| < \frac{1}{2}\varepsilon \text{ for all } n > N.$$

$$\text{Now } |s_m - s_n| = |(s_m - s) + (s - s_n)| \leq |s_m - s| + |s_n - s|$$
$$< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \text{ for all } m, n > N,$$

which proves the proposition.

Is the converse true ? Is every Caucny sequence a convergent sequence ? Consider the following example.

Suppose we are concerned with the open interval (0, 1) only. The sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ lies in this interval. This is a Cauchy sequence. The limit of this sequence is 0, which does not lie within (0, 1). So, as far as this interval is concerned, there is no number s which satisfies the property of § 6.1. Hence the sequence is not convergent.

Another example is provided by the space * of rational numbers. Take the sequence

$$1.4, 1.41, 1.414, 1.4142, \dots$$

in which the nth term is an n decimal place approximation to $\sqrt{2}$. This is a Cauchy sequence, but not a convergent sequence in the space of rationals (since the limit $\sqrt{2}$ does not lie in this space).

We define a complete space as the space in which every Cauchy sequence has a limit. The space of real numbers is a complete space. For such a space i.e. ***for real numbers, every***

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* A set of numbers on which the idea of distance is defined is called space.

Cauchy sequence is convergent. For a complete space we can give the definition :

A sequence $\{s_n\}$ *is convergent if for any positive number* ε *however small, we can find a number N such that*

$$|s_m - s_n| < \varepsilon \text{ for all } m, n > N.$$

This is known as **Cauchy's criterion of convergence**. i.e. a sequence converges if and only if it is a Cauchy sequence.

6.2. Infinite Series. Let u_n be a function of n, which has a definite value for all positive integral values of n. Then an expression of the form

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots,$$

in which every term is followed by another term, is called an ***infinite series***. The above series is denoted by

$$\sum_{n=1}^{\infty} u_n \quad \text{or} \quad \sum u_n,$$

and the sum of its first n terms is denoted by s_n.

An example of an infinite series is the geometric series

$$1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \ldots \qquad (1)$$

Let us consider the sum of its first n terms. For this series

$$s_n = 2 - \left(\tfrac{1}{2}\right)^{n-1} \quad \text{or} \quad 2 - s_n = \left(\tfrac{1}{2}\right)^{n-1}.$$

We see that when n gets larger and larger, $\left(\tfrac{1}{2}\right)^{n-1}$ become smaller and smaller. By choosing n sufficiently large, $\left(\tfrac{1}{2}\right)^{n-1}$ can be made as small as we please. Therefore s_n approaches the limit 2 as n tends to infinity. A series like (1), for which s_n tends to a finite limit, is called a ***convergent series***.

Now consider the infinite series

$$1 + 2 + 2^2 + 2^3 + \ldots \qquad (2)$$

In this case $s_n = 2^n - 1$. As n increases, s_n also increases; and by choosing n sufficiently large, s_n can be made to exceed any given number, however large that number may be. Thus s_n tends to infinity as n tends to infinity. A series like (2) is called a ***divergent series***.

There is a third type of series: consider for example

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots,$$

or

$$1 - 3 + 5 - 7 + 9 - 11 + \ldots.$$

In the first series s_n is equal to 1 or 0, according as n is odd or even. In the second series s_n is alternately positive and negative, while numerically it goes on increasing as n increases. Such series, in which the sum to n terms fluctuates from one value to another, or from a positive value to a negative value, and does not approach a limit, are called ***oscillatory series***.

Rules of algebra, like addition, multiplication, etc. cannot be always applied to infinite series. For example, denoting the series (2) by S, and writing

$$S = 1 + 2 + 4 + 8 + 16 + \ldots,$$

we have

$$2S = 2 + 4 + 8 + 16 + \ldots.$$

By subtraction we get $S = -1$, which is absurd, since all the terms in S are positive. It can be shown that the addition of two infinite series is valid for convergent series only; and their multiplication is valid for a particular class of convergent series. For this reason, it is of importance to find out whether a given series is convergent or not.

We frequently make use of infinite series in calculus. For example, the exponential function e^x is defined as the 'sum' of the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Unless we can assign a definite meaning to the sum of an infinite series, it will not be of any use to deal with such a function. We now proceed to do this.

6.21. Definition of Convergence. The concept of limits introduced in Chapter II can be utilised to give an exact meaning to the convergence and sum of an infinite series.

The partial sums $s_1, s_2, \ldots, s_n, \ldots$ in fact from a sequence, and the same ideas can be used. However, we shall give an independent definition.

An infinite series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is said to be **CONVERGENT** *if s_n, the sum of its first n terms, tends to a finite limit s as n tends to infinity.* We call s the **SUM** of the series, and write

$$s = \lim_{n \to \infty} s_n.$$

If s_n tends to plus or minus infinity as n tends to infinity, the series is said to be ***divergent.***

If s_n does not tend to a finite limit, or to plus or minus infinity, the series is said to be ***oscillatory***.

In the last case the value of s_n fluctuates, either within a finite rage, or the numerical value of s_n tends to infinity while its sign is alternately positive and negative. In the former case the series is said to *oscillate finitely*; and in the latter case it is said to *oscillate infinitely*.

Series which diverge or oscillate are often said to be non-convergent. *

Recalling the definition of the limit, we see that *an infinite series $\sum u_n$ converges to a sum s if for any given* ***positive number*** *ε* ***however small, we can find a positive number N, such that***

$$|s_n - s| < \varepsilon, \quad \textit{for all } n > N,$$

where s_n is the sum of the first n terms of $\sum u_n$.

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* The student should note that the words "divergent" and "oscillatory" are used differently by different writers. Some writers regard "divergent" as equivalent to "non-convergent", while some call a series "oscillatory" only if it oscillates finitely, series which oscillate infinitely being classed as "divergent". But the classification given in this book is the one generally used.

As all the limits for infinite series are invariably taken as $n \to \infty$, we shall write in this chapter " $\lim_{n\to\infty}$" simply as "lim".

Ex. Discuss the convergence of the series

$$a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} + \ldots \qquad (1)$$

The terms are in geometric progression with common ratio x. The sum of the first n terms is

$$s_n = \frac{a(1-x^n)}{1-x}.$$

If $|x|$, the absolute value of x, is less than 1, then $|x^n|$ decreases continually as n increases and tends to zero as $n \to \infty$. Hence

$$\lim s_n = \frac{a}{1-x}$$

So for $|x| < 1$, the series (1) is convergent.

If $|x| > 1$, $|x^n|$ continually increases with n and tends to infinity as $n \to \infty$. The sign of x^n remains always positive when x is positive, but is alternately positive and negative when x is negative. Hence the series diverges for $x > 1$, and oscillates infinitely for $x < -1$.

When $x = 1$, the series becomes

$$a + a + a + \ldots + a + \ldots,$$

which is divergent, since $s_n = na$ and tends to infinity as $n \to \infty$.

When $x = -1$, the series becomes

$$a - a + a - a + \ldots$$

Here s_n is alternately a and 0, and does not tend to a limit as $n \to \infty$. Hence the series is oscillatory.

EXAMPLES

Find the sum of the first n terms of the following series and determine whether the series are convergent or divergent.

1. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots$

2. $16-12+9-\frac{27}{4}+\frac{81}{16}-\ldots.$
3. $1+2+3+4+\ldots$
4. $1+2x+3x^2+4x^3+\ldots,\quad \text{if } |x|<1.$
5. $\dfrac{1}{(m+1)(m+2)}+\dfrac{1}{(m+2)(m+3)}+\dfrac{1}{(m+3)(m+4)}+\ldots.$
6. $\log_e 2+\log_e \frac{3}{2}+\log_e \frac{4}{3}+\log_e \frac{5}{4}+\ldots$ [*Kanpur* 1995]

ANSWERS

1. Convergent. 2. Convergent. 3. Divergent.
4. Convergent. 5. Convergent. 6. Divergent.

SERIES OF POSITIVE TERMS

6.3. Some general theorems. We cannot find the sum of the first *n* terms of every series. Therefore the definition of convergence cannot be applied directly in every case and we have to devise other methods for testing the convergence of a given series.

The following general theorems on infinite series will be found useful in obtaining such methods. The last two theorems are for series all of whose terms are positive. Tests of convergence for such series are easier to obtain. We shall see later that quite often the convergence of a series of positive and negative terms also depends on the convergence of a series of positive terms.

THEOREM I. ***For every convergent series*** $\sum u_n$

$$\lim u_n = 0.$$

For $u_n = s_n - s_{n-1}$. On taking the limits as $n\to\infty$, s_n and s_{n-1} both tend to the same limit s. Therefore $u_n \to 0$ as $n\to\infty$.

From this theorem we can devise the following test :

If $\lim u_n \neq 0$, ***the series is non-convergent.***

If in addition the terms of $\sum u_n$ are all of the same sign, the *series is divergent*.

The student should note that the converse of the above theorem is not true. If $\lim u_n = 0$, the series may or may not be convergent.

For example, consider the series

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots+\frac{1}{\sqrt{n}}+\ldots.$$

Since the terms go on decreasing,

$$s_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ldots+\frac{1}{\sqrt{n}}.$$

i.e. $s_n > \sqrt{n}$, which tends to infinity as n tends to infinity. Therefore the series is divergent, even though $\lim u_n = 0$.

Thus, in order that $\sum u_n$ *be convergent, it is necessary but not sufficient that*

$$\lim u_n = 0.$$

THEOREM II. ***The nature of an infinite series remains unaltered by the addition or removal of a finite number of terms, or by the multiplication of each term by a fixed number (≠ 0).***

Let $u_1+u_2+\ldots$ be a given series, and let $a+u_1+u_2+\ldots$ be a series obtained by adding one more term to it. Denote by s_n the sum of the first n term of the first series, and by S_n the sum of the first n terms of the second series. Then

$$S_n = a+s_{n-1}$$

so that $\qquad \lim S_n = a+\lim s_{n-1}.$

If s_n tends to a finite limit s as $n\to\infty$, S_n tends to $a+s$. If $s_n\to\infty$ as $n\to\infty$, S_n also tends to infinity. If s_n oscillates, so does S_n. Hence the nature of the series remains unaltered by the addition of one term. A similar proof shows that it remains unaltered by the addition of a *finite* number of terms.

If $a = -u_1$, the second series becomes $u_2+u_3+\ldots.$ So the theorem is true for the removal of terms also.

If each term of $u_1 + u_2 + \dots$ is multiplied by a non-zero number k, we get the series $ku_1 + ku_2 + \dots$. If σ_n denotes the sum of the first n terms of the series, $\sigma_n = ks_n$. Therefore

$$\lim \sigma_n = k \lim s_n .$$

Hence the nature of the second series is the same as that of first.

THEOREM III. ***A series of positive terms is convergent if s_n, the sum to n terms, is less than a fixed number for all values of n.***

Let the series be $u_1 + u_2 + u_3 + \dots$, where $u_1, u_2, \dots$ are all positive. Then s_n, i.e. $u_1, u_2 + \dots + u_n$, will go on increasing as n increases, and may tend to a finite limit or to plus infinity. It cannot oscillate.

Therefore, if s_n remains less than a fixed number for all values of n, it cannot tend to infinity, and so must tend to a finite limit. Hence the given series is convergent.

THEOREM IV. ***If each term of a series $\sum u_n$ of positive terms, does not exceed the corresponding term of a convergent series $\sum v_n$ of positive terms, then $\sum u_n$ is convergent.***

If, on the contrary, each term of $\sum u_n$ exceeds (or equals) the corresponding term of a divergent series of positive terms, then $\sum u_n$ is divergent.

For the first part we are given

$$u_n \le v_n , \qquad (1)$$

for all n. Denoting by s_n and σ_n the sum to n terms of $\sum u_n$ and $\sum v_n$ respectively, we see therefore that

$$s_n \le \sigma_n .$$

But $\sum v_n$ is a convergent series of positive terms, having the sum σ, say. Therefore $\sigma_v < \sigma$. It follows that

$$s_n < \sigma$$

for all values of n. Hence, by theorem III, $\sum u_n$ is convergent.

For the second part, we are given

$$u_n \ge v_n \qquad (2)$$

for all n. Therefore

$$s_n \ge \sigma_n .$$

Since $\sum v_n$ is divergent, $\sigma_n \to \infty$ as $n \to \infty$. Hence s_n also tends to infinity as $n \to \infty$. So $\sum u_n$ is divergent.

By theorem II we see that the above theorem is true even if the inequality (1) or (2) holds after a finite number of terms and not from the very beginning.

Ex. 1. Show that the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$ is divergent.

For this series $\lim u_n = \lim \left(1 + \frac{1}{n}\right)^n = e \ne 0.$

Hence the series is divergent.

Ex. 2. Show that the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ is convergent.

For this series $s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$

$$< 1 + \frac{1}{12} + \frac{1}{23} + \dots + \frac{1}{(n-1)n},$$

i.e. $$< 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right),$$

i.e. $$< 2 - 1/n$$

showing that s_n is less than 2 for all values of n.

Therefore the series is convergent.

6.4. Ratio Test (or D'Alembert's Test *). *An infinite series of positive terms is convergent if from and after some term the ratio of each term to the preceding term is less than a fixed number which is less than unity.*

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* Called after the French mathematician Jean-le-Rond D'Alembert (1717-1783). He is more famous for his researches in dynamics.

The series is divergent if the above ratio is greater than or equal to unity.

(i) Let the series beginning from the specified term be denoted by

$$u_1 + u_2 + u_3 + u_4 + \ldots$$

and let $\dfrac{u_2}{u_1} < r,\ \dfrac{u_3}{u_2} < r,\ \dfrac{u_4}{u_3} < r, \ldots,$ where $r < 1$.

Then
$$u_n = \frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \ldots \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} \cdot u_1$$
$$< r \cdot r \ldots . r \cdot r \cdot u_1, \text{ i.e. } u_1 r^{n-1}.$$

Hence $u_1 + u_2 + u_3 + \ldots$ is term by term less than the series

$$u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1} + \ldots . \qquad (1)$$

Series (1) is convergent since $r < 1$ (Ex. § 6.21). Therefore the given series is also convergent.

(ii) Again, if $\dfrac{u_2}{u_1} > r,\ \dfrac{u_3}{u_2} > r,\ \dfrac{u_4}{u_3} > r, \ldots,$

where $r \geq 1$, then the given series is term by term greater than the divergent series

$$u_1 + u_1 r + u_1 r^2 + \ldots .$$

Hence it is divergent.

COROLLARY. *The series $\sum u_n$ is convergent if*

$$\lim \frac{u_{n+1}}{u_n} < 1,$$

and divergent if $\lim u_{n+1} / u_n > 1$.

Let $\lim u_{n+1} / u_n$ be l, and consider first the case when $l < 1$. Choose a positive number ε, such that $l + \varepsilon < 1$.

Then we can find a number N such that

$$\frac{u_{n+1}}{u_n} - l < \varepsilon$$

for all $n > N$. Therefore, for all except a finite number of terms,

$$\frac{u_{n+1}}{u_n} < l + \varepsilon$$

a fixed number less than unity. Hence the series is convergent.

The proof when $l > 1$ is similar.

When $\lim u_{n+1} / u_n = 1$, the test fails; and we cannot say whether the series is convergent or divergent.

NOTE. The student often wonders why it is necessary that u_{n+1} / u_n should be less than a fixed number which is less than unity. He thinks that this statement is equivalent to saying that u_{n+1} / u_n is always less than unity. But this is not so. If u_{n+1} / u_n is always less than unity, the series may or may not be convergent.

For example in the two series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \text{ and } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots,$$

u_{n+1} / u_n is equal to $\sqrt{n/(n+1)}$ and $[n/(n+1)]^2$ respectively, both of which are less than 1. But for both of these there is no fixed number l such that

$$u_{n+1} / u_n < l < 1$$

for every value of n. Now the first of these two series is divergent while the second is convergent (§ 6.3), which shows that it is not sufficient for convergence that u_{n+1} / u_n be less than unity. It may be noted that for both the above series $\lim u_{n+1} / u_n = 1$, and so the results are not contrary to the corollary.

In applying the ratio test, it is more convenient to find $\lim u_{n+1} / u_n$, *than to find a number less than 1 and exceeding each ratio. If this limit is unity, then further tests should be applied.*

Ex. 1. Test the convergence of the series

$$\frac{x}{1 \cdot 2} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{5 \cdot 6} + \frac{x^4}{7 \cdot 8} + \ldots, \text{ where } x > 0.$$

[*Avadh* 08, *Kashi Vidyapeeth* 09]

Here $u_n = x^n / (2n-1)2n$ and $u_{n+1} = x^{n+1} / (2n+1)(2n+2)$.

Therefore $\lim \dfrac{u_{n+1}}{u_n} = \lim \dfrac{(2n-1)\,2n}{(2n+1)(2n+2)}\,x$

$$= \lim \frac{1 + 1/2n}{(1+1/2n)(1+1/2n)}\,x = x.$$

Hence if $x < 1$, the series is convergent and if $x > 1$, the series is divergent. If $x = 1$, $\lim u_{n+1}/u_n = 1$ and further tests are necessary. [It will be shown later the series is convergent in this case.]

Ex. 2. Find whether the series

$$2 + \tfrac{3}{2}x + \tfrac{4}{3}x^2 + \tfrac{5}{4}x^3 + \ldots$$

is convergent or divergent for positive values of x.

We have $u_n = (n+1)x^{n-1}/n$ and $u_{n+1} = (n+2)x^n/(n+1)$.

Therefore $\lim \dfrac{u_{n+1}}{u_n} = \lim \dfrac{n(n+2)}{(n+1)^2}\,x = x$.

Hence if $x < 1$, the series is convergent; and if $x > 1$, the series is divergent. If $x = 1$, the series becomes $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \ldots$ which is divergent since

$$\lim u_n = \lim \frac{n+1}{n} = 1 \neq 0.$$

6.41. Root Test. *A series $\sum u_n$ of positive terms is convergent if for every value of n, $\sqrt[n]{u_n}$ is less than a fixed number which is less than unity.*

The series is divergent if $\sqrt[n]{u_n} \geq 1$ for every value of n.

(i) Let $\sqrt[n]{u_n} < r$ where $r < 1$.

Then $u_n < r^n$ for all values of n. Therefore

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots < r + r^2 + r^3 + \ldots + r^n + \ldots,$$

---

that is $\sum u_n < r/(1-r)$ which is a fixed number.

Hence $\sum u_n$ is convergent.

(ii) If $\sqrt[n]{u_n} \geq 1$, then $u_n \geq 1$ for all values of n. Hence the series is divergent.

NOTE. The theorem is true even if the condition holds from and after a fixed term.

COROLLARY. The series $\sum u_n$ is convergent if $\lim \sqrt[n]{u_n} < 1$, and divergent if $\lim \sqrt[n]{u_n} \geq 1$,

If $\lim \sqrt[n]{u_n} = 1$, the test fails.

The proof is similar to that of the corollary in the last article.

Ex. Test for convergence the series

$$1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots, \quad \text{where } x > 0$$ [*Kanpur* 1994]

Consider the series obtained on omitting the first term; for it

$$u_n = x^n/(n+1)^n,$$

so that $\lim \sqrt[n]{u_n} = \lim x/(n+1) = 0$.

Hence the series is convergent for all values of x.

EXAMPLES

Determine whether the following series are convergent or divergent.

1. $\dfrac{1}{\sqrt{2}} + \dfrac{2}{\sqrt{5}} + \dfrac{4}{\sqrt{17}} + \dfrac{8}{\sqrt{65}} + \ldots + \dfrac{2^n}{\sqrt{4^n+1}} + \ldots$

2. $\dfrac{1}{1+2^{-1}} + \dfrac{2}{1+2^{-2}} + \dfrac{3}{1+2^{-3}} + \ldots.$

3. $\dfrac{1}{1+2} + \dfrac{2}{1+2^2} + \dfrac{3}{1+2^3} + \ldots.$

4. $1 + \dfrac{3}{2!} + \dfrac{5}{3!} + \dfrac{7}{4!} + \ldots.$

5. $1 + \dfrac{2p}{2!} + \dfrac{3p}{3!} + \dfrac{4p}{4!} + \ldots.$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(1+1/n)^n}$

7. $\displaystyle\sum \sqrt{\frac{2^n-1}{3^n-1}}$

Test for convergence the following series for positive value of x.

8. $1+3x+5x^2+7x^3+\ldots.$

9. $1^2+2^2x+3^2x^2+4^2x^3+\ldots.$

10. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots.+\frac{n^2-1}{n^2+1}x^n+\ldots.$ [*Indore* 2005]

11. $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots.+\frac{2^n-2}{2^n+1}x^{n-1}+\ldots.$

12. $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\ldots.$

ANSWERS

1. Divergent **2.** Divergent. **3.** Convergent.

4. Convergent. **5.** Convergent. **6.** Divergent.

7. Convergent. **8.** Con. if $x<1$, div. if $x \ge 1$.

9. Con. if $x<1$, div. if $x \ge 1$. **10.** Con. if $x<1$, div. if $x \ge 1$.

11. Con. if $x<1$, div. if $x \ge 1$. **12.** Con. if $x<1$, div. if $x \ge 1$.

6.5. The series $\sum 1/n^p$. *The infinite series*

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$

is convergent if $p>1$, and divergent if $p \le 1$.

[*Meerut* 05, *Indore* 05, *Himachal* 06, Patna 07, *Jabalpur* 08, *Bilaspur* 09, *Kashi Vidyapeeth* 10, *Raipur* 13, *Allahabad* 13]

CASE I. Let $p>1$. As the terms are all positive we can group them as we like. Now group them as follows :

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)$$
$$+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots \qquad (1)$$

This series is less, term by term, than the series

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)$$
$$+\left(\frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p}\right)+\ldots \qquad (2)$$

But (2) can be written as

$$1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots$$

which is a geometric series with the common ratio $2/2^p=1/2^{p-1}$. So the series (2) is convergent since $1/2^{p-1}<1$. Therefore the series (1) is also convergent.

CASE II. Let $p=1$. Then the series becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots.$ The terms in this series can be grouped as

$$1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\left(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\right)+\left(\tfrac{1}{9}+\ldots+\tfrac{1}{16}\right)+\ldots. \qquad (3)$$

This is, term by term, greater than

$$1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)+\left(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}\right)+\left(\tfrac{1}{16}+\ldots+\tfrac{1}{16}\right)+\ldots.$$

i.e.
$$1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\ldots.$$

which is divergent. Therefore (3) is also divergent.

CASE III. Let $p<1$ (negative values of p being included). Then the series is term by term greater than the series (3), and is, therefore, divergent.

The series $\sum 1/n^p$ will be found useful as an auxiliary series for the purpose of the comparison test given below.

6.51. Comparison Test. *If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $\lim (u_n/v_n)$ is finite and non-zero, then the two series are either both convergent or both divergent.*

Let $\lim u_n/v_n=l$. Choose a small positive number ε $(<l)$: Then we can find a number N such that

$$|u_n/v_n-l|<\varepsilon$$

for all $n > N$. Therefore, for $n > N$,

$$u_n / v_n < l + \varepsilon \tag{1}$$

and

$$u_n / v_n > l - \varepsilon. \tag{2}$$

(i) If $\sum v_n$ is convergent, $\sum (l + \varepsilon) v_n$ is also convergent. Again by (1), $\sum u_n$ is term by term less than $\sum (l + \varepsilon) u_n$ except for a finite number of terms. Therefore $\sum u_n$ is also convergent.

(ii) If $\sum v_n$ is divergent, so is $\sum (l - \varepsilon) v_n$. Also, by (2), $\sum u_n$ is term by term greater than $\sum (l - \varepsilon) u_n$ except for a finite number of terms. Therefore $\sum u_n$ is divergent.

To apply the comparison test to a given series $\sum u_n$, the auxiliary series $\sum v_n$ must be chosen in such a way that $\lim u_n / v_n$ is non-zero and finite. This can generally be done by taking v_n equal to the term of the highest degree in n (or the lowest degree in $1/n$) occurring in u_n.

In simple terms, applying the comparison test means that if the " larger " series converges, so does the " smaller " one and if the " smaller " series diverges, so does the " larger " one.

Ex. 1. Test for convergence $.\sum_{n=1}^{\infty} [\sqrt{n^2 + 1} - n]$ **[*Jabalpur* 09]**

Here $u_n = \sqrt{n^2 + 1)} - n = n(1 + 1/n^2)^{1/2} - n$

$$= n\left\{1 + \tfrac{1}{2}(1/n^2) - \tfrac{1}{8}(1/n^4) + \ldots\right\} - n$$

$$= \tfrac{1}{2}(1/n) - \tfrac{1}{8}(1/n^3) + \ldots.$$

Take $v_n = 1/n$, then

$$\lim u_n / v_n = \lim\{\tfrac{1}{2} - \tfrac{1}{8}(1/n^2) + \ldots\} = \tfrac{1}{2}$$

which is finite and non-zero. Since $\sum v_n$ is divergent, therefore $\sum u_n$ is also divergent.

Ex. 2. Test the convergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots \text{ to } \infty.$$ [*Lucknow* 1995]

Here $u_n = x^{2n-2}/(n+1)\sqrt{n}$ and $u_{n+1} = x^{2n}/(n+2)\sqrt{n+1}$.

Therefore $\lim \dfrac{u_{n+1}}{u_n} = \lim \dfrac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}} x^2 = x^2$.

Hence (by the Ratio test), the series is convergent if $x^2 < 1$, and divergent if $x^2 > 1$.

If $x^2 = 1$, then $u_n = 1/(n+1)\sqrt{n}$ and taking $v_n = 1/n^{3/2}$, we find that $\lim u_n / v_n = 1$.

Since $\sum v_n$ is convergent, the given series also is convergent for $x^2 = 1$.

EXAMPLES

Determine whether the following series are convergent or divergent.

1. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots.$
2. $\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \ldots.$
3. $\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \ldots.$

[*Jabalpur* 07, *Himachal* 07, *Indore* 08, *Avadh* 11]

4. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots.$ **[*Meerut* 2004]**
5. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots.$ **[*Garhwal* 06, 10, *Avadh* 10]**

Test the convergence of the series whose general term is:

6. $\dfrac{\sqrt{n}}{n^2 + 1}$.
7. $\dfrac{n}{(a + nb)^2}$
8. $(n^3 + 1)^{1/3} - n$. **[*Garhwal* 07, 12]**
9. $\sqrt{n^3 + 1} - \sqrt{n^3 - 1}$. **[*Kashi Vidyapeeth* 2008]**
10. $\sqrt{n^4 + 1} - \sqrt{n^4 - 1}$.

[*Meerut* 03, 05, *Allahabad* 05, *Avadh* 06, *Patna* 07, *Bilaspur* 07]

Test the convergence of the following series for $x > 0$

11. $1 + \frac{1}{2}x + \frac{1}{5}x^2 + \frac{1}{10}x^3 + \ldots + x^n/(n^2 + 1) + \ldots.$

12. $\frac{x}{1\cdot2}+\frac{x^2}{2\cdot3}+\frac{x^3}{3\cdot4}+\frac{x^4}{4\cdot5}+\dots$ [*Kashi Vidyapeeth* 2010]

13. $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\dots$

14. $\frac{x}{a\cdot1^2+b}+\frac{2x^2}{a\cdot2^2+b}+\dots+\frac{nx^n}{a\cdot n^2+b}+\dots$

ANSWERS

1. Divergent.
2. Divergent.
3. Convergent.
4. Divergent.
5. Con. if $p>2$, div. if $p\le 2$.
6. Convergent.
7. Divergent.
8. Convergent.
9. Convergent.
10. Convergent.
11. Con. if $x\le 1$, div. if $x>1$
12. Con. if $x\le 1$, div. if $x>1$.
13. Con. if $x<1$, div. if $x\ge 1$.
14. Con. if $x<1$, div. if $x\ge 1$.

6.6. Comparison of ratios. *If* $\sum u_n$ *and* $\sum v_n$ *be two series of positive terms, then* $\sum u_n$ *is convergent if (i)* $\sum v_n$ *is convergent, and (ii) from and after some particular term*

$$\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}.$$

$\sum u_n$ *is divergent, if (i)* $\sum v_n$ *is divergent, and (ii) from and after some particular term*

$$\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}.$$

Since the removal of a finite number of terms does not alter the nature of the series, we shall suppose that the inequality is satisfied from the very first term.

I. Let $\frac{u_2}{u_1} < \frac{v_2}{v_1},\ \frac{u_3}{u_2} < \frac{v_3}{v_2},\dots$

Then $u_1+u_2+u_3+\dots+u_n = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\dots\right)$

$$< u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\dots\right),$$

i.e. $$<\frac{u_1}{v_1}\left(v_1+v_2+v_3+\dots+v_n\right).$$

Therefore, if $\sum v_n$ is convergent, $\sum u_n$ is also convergent.

II. Let $\frac{u_2}{u_1} > \frac{v_2}{v_1},\ \frac{u_3}{u_2} > \frac{v_3}{v_2},\dots$

Then $u_1+u_2+u_3+\dots+u_n = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\dots\right)$

$$> u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\dots\right),$$

i.e. $$>\frac{u_1}{v_1}\left(v_1+v_2+v_3+\dots+v_n\right).$$

Therefore, if $\sum v_n$ is divergent, $\sum u_n$ is also divergent.

NOTE. The test for convergence given in the article above can also be stated as follows :

$\sum u_n$ is convergent if (i) $\sum v_n$ is convergent, and (ii) from and after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}.$$

This form of inequality is often more convenient to apply, as in the articles 6.61 and 6.62 below. The test for divergence can be stated similarly.

By taking $\sum 1/n^p$ for $\sum v_n$ and applying the above test, we can derive tests which are very convenient for determining convergence when $\lim u_{n+1}/u_n$ is unity.

6.61. Higher Ratio Test (or Raabe's Test). *A series $\sum u_n$ of positive terms is convergent or divergent according as*

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)>1 \text{ or } <1.$$

Let $$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=l$$

where $l>1$. Choose a number p lying between l and 1 and compare the given series $\sum u_n$ with the auxiliary series $\sum 1/n^p$. This series is convergent since $p>1$.

Therefore, $\sum u_n$ will be convergent if from and after some particular term

$$\frac{u_n}{u_{n+1}}>\frac{(n+1)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p$$

i.e. if $$\frac{u_n}{u_{n+1}}>1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots$$

or $$n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{2n}+\ldots. \qquad (1)$$

By taking n sufficiently large, the left-hand side can be made to approach 1 as nearly as we please, and the right-hand side can be made to approach p as nearly as we please. Also, l is greater than p. Therefore for all sufficiently large values of n, the inequality (1) is satisfied. Hence $\sum u_n$ is convergent.

The other case, when $l<1$ can be proved similarly.

NOTE 1. The above test should be applied only in those cases in which $\lim u_{n+1}/u_n=1$ and the simple ratio test (§ 6.4) fails. When the present test also fails, further tests are necessary. It will be shown in § 6.73, however, that in such cases the series is divergent, provided that u_n/u_{n+1} does not involve n as a logarithm or an exponent.

NOTE 2. The student should note that for the simple ratio test the limit of u_{n+1}/u_{n} was determined, while the present and all the subsequent tests (§§ 6.62, 6.72, 6.73) employ the ratio u_n/u_{n+1}

Ex. Test for convergence the series

$$1+\frac{3}{7}x+\frac{3\cdot6}{7\cdot10}x^2+\frac{3\cdot6\cdot9}{7\cdot10\cdot13}x^3+\frac{3\cdot6\cdot9\cdot12}{7\cdot10\cdot13\cdot16}x^4+\ldots.$$

Here $$u_n=\frac{3\cdot6\cdot9\ldots(3n-3)}{7\cdot10\cdot13\ldots(3n+1)}x^{n-1}$$

and $$u_{n+1}=\frac{3\cdot6\cdot9\ldots(3n-3)3n}{7\cdot10\cdot13\ldots(3n+1)(3n+4)}x^n.$$

Therefore $$\frac{u_{n+1}}{u_n}=\frac{3n}{3n+4}x \text{ and } \lim\frac{u_{n+1}}{u_n}=x.$$

Hence the series is convergent if $x<1$ and divergent if $x>1$ (considering only positive values of x). If $x=1$,

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left(\frac{3n+4}{3n}-1\right)=\frac{4}{3}$$

which is greater than 1. Hence the series is convergent.

6.62. Logarithmic Ratio Test. *A series of positive terms is convergent or divergent according as*

$$\lim\left\{n\log\frac{u_n}{u_{n+1}}\right\}>1 \text{ or } <1.$$

Let $$\lim n\log\left(u_n/u_{n+1}\right)=l,$$

where $l>1$, and choose a number p lying between l and 1. Then, comparing the given series $\sum u_n$ with the auxiliary series $\sum 1/n^p$, which is convergent since $p>1$, we see that $\sum u_n$ is also convergent if from and after some particular term

$$\frac{u_n}{u_{n+1}}>\frac{(1+n)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p$$

Taking the logarithms of both the sides,

if $$\log \frac{u_n}{u_{n+1}} > p \log\left(1+\frac{1}{n}\right),$$

i.e. if $$\log \frac{u_n}{u_{n+1}} > p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)$$

or $$n \log \frac{u_n}{u_{n+1}} > p+\frac{p}{n}\left(-\frac{1}{2}+\frac{1}{3n}-\ldots\right)$$

By taking n sufficiently large, the left-hand side can be made to approach l as nearly as we please, and the right-hand side can be made to approach p as nearly as we please. Also, $l > p$. Therefore the above inequality is satisfied for all sufficiently large values of n. Hence $\sum u_n$ is convergent.

The other case, when $l < 1$, can be proved similarly.

This test is an alternative to Raabe's test and should be applied when $\lim u_n / u_{n+1} = 1$, and taking the logarithm of u_n / u_{n+1} makes the evaluation of the limit easier.

Ex. Test the convergence of the following series

$$1+\frac{2x}{2!}+\frac{3^2 x^2}{3!}+\frac{4^3 x^3}{4!}+\frac{5^4 x^4}{5!}+\ldots.$$

[*Himachal* 06, *Rohilkhand* 10, *Garhwal* 11]

Here $$u_n = \frac{n^{n-1} x^{n-1}}{n!} \text{ and } u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

Therefore $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^n}{n^{n-1}} \cdot \frac{x}{n+1} = \left(1+\frac{1}{n}\right)^n \cdot \frac{n}{n+1} x$$

and $$\lim u_n / u_{n+1} = ex.$$

Hence the series is convergent if $ex < 1$ i.e. if $x < 1/e$; and divergent if $x > 1/e$. If $x = 1/e$, we have

$$\lim n \log \frac{u_n}{u_{n+1}} = \lim n \log\left\{\frac{e}{(1+1/n)^{n-1}}\right\}$$

$$= \lim n\,\{\log e - (n-1)\log(1+1/n)\}$$

$$= \lim n\left\{1-\frac{n-1}{n}+\frac{n-1}{2n^2}-\frac{n-1}{3n^3}+\ldots\right\}$$

$$= \lim\left\{n-(n-1)+(\frac{1}{2}-1/2n)-(n-1)/3n^2+\ldots\right\}$$

$$= 1+\frac{1}{2}=\frac{3}{2}$$

which is greater than 1. Hence the series is convergent for $x = 1/e$.

EXAMPLES

Find whether the following series are convergent or divergent.

1. $1+a+\frac{a(a+1)}{1.2}+\frac{a(a+1)(a+2)}{1.2.3}+\ldots.$ **[*Kashi Vidyapeeth* 2012]**

2. $\frac{1^2}{4^2}+\frac{1^2\cdot 5^2}{4^2\cdot 8^2}+\frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+\ldots.$ **[*Himachal* 2007]**

Test for $x > 0$, the convergence of the following series.

3. $1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^4}{8}+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{x^6}{12}+\ldots.$ **[*Rohilkhand* 2008]**

4. $x^2+\frac{2^2}{3\cdot 4}x^4+\frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6+\frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8+\ldots.$ **[*Bundelkhand* 06, *Bilaspur* 07, *Garhwal* 09, *Raipur* 11, 14, *Allahabad* 14]**

5. $1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots.$ **[*Raipur* 2015]**

6. $x+\frac{2^2 x^2}{2!}+\frac{3^3 x^3}{3!}+\frac{4^4 x^4}{4!}+\frac{5^5 x^5}{5!}+\ldots.$

[*Himachal* 05, *Indore* 06, *Bilaspur* 06, *Jabalpur* 07, *Garhwal* 09, *Avadh* 11, *Allahabad* 13]

ANSWERS

1. Con. if $a \le 0$, div. if $a > 0$. **2.** Convergent

3. Con. if $x^2 \le 1$, div. if $x^2 > 1$. **4.** Con. if $x^2 \le 1$, div. if $x^2 > 1$.

5. Con. if $x < 1$, div. if $x \ge 1$. **6.** Con. if $x < 1/e$, div. if $x \ge 1/e$.

Taking the logarithms of both the sides,

if $$\log \frac{u_n}{u_{n+1}} > p \log\left(1+\frac{1}{n}\right),$$

i.e. if $$\log \frac{u_n}{u_{n+1}} > p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)$$

or $$n \log \frac{u_n}{u_{n+1}} > p + \frac{p}{n}\left(-\frac{1}{2}+\frac{1}{3n}-\ldots\right)$$

By taking n sufficiently large, the left-hand side can be made to approach l as nearly as we please, and the right-hand side can be made to approach p as nearly as we please. Also, $l > p$. Therefore the above inequality is satisfied for all sufficiently large values of n. Hence $\sum u_n$ is convergent.

The other case, when $l < 1$, can be proved similarly.

This test is an alternative to Raabe's test and should be applied when $\lim u_n / u_{n+1} = 1$, and taking the logarithm of u_n / u_{n+1} makes the evaluation of the limit easier.

Ex. Test the convergence of the following series

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\frac{5^4x^4}{5!}+\ldots.$$

[*Himachal* **06**, *Rohilkhand* **10**, *Garhwal* **11**]

Here $$u_n = \frac{n^{n-1}x^{n-1}}{n!} \text{ and } u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

Therefore $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^n}{n^{n-1}} \cdot \frac{x}{n+1} = \left(1+\frac{1}{n}\right)^n \cdot \frac{n}{n+1} x$$

and $$\lim u_n / u_{n+1} = ex.$$

Hence the series is convergent if $ex < 1$ i.e. if $x < 1/e$; and divergent if $x > 1/e$. If $x = 1/e$, we have

$$\lim n \log \frac{u_n}{u_{n+1}} = \lim n \log\left\{\frac{e}{(1+1/n)^{n-1}}\right\}$$

$$= \lim n\,\{\log e - (n-1)\log(1+1/n)\}$$

$$= \lim n\left\{1-\frac{n-1}{n}+\frac{n-1}{2n^2}-\frac{n-1}{3n^3}+\ldots\right\}$$

$$= \lim\left\{n-(n-1)+(\tfrac{1}{2}-1/2n)-(n-1)/3n^2+\ldots\right\}$$

$$= 1+\frac{1}{2} = \frac{3}{2}$$

which is greater than 1. Hence the series is convergent for $x = 1/e$.

EXAMPLES

Find whether the following series are convergent or divergent.

1. $1+a+\dfrac{a(a+1)}{12}+\dfrac{a(a+1)(a+2)}{123}+\ldots.$

[*Kashi Vidyapeeth* **2012**]

2. $\dfrac{1^2}{4^2}+\dfrac{1^2\cdot 5^2}{4^2\cdot 8^2}+\dfrac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+\ldots.$ [*Himachal* **2007**]

Test for $x > 0$, the convergence of the following series.

3. $1+\dfrac{1}{2}\cdot\dfrac{x^2}{4}+\dfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\dfrac{x^4}{8}+\dfrac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}\cdot\dfrac{x^6}{12}+\ldots$.[*Rohilkhand* **2008**]

4. $x^2+\dfrac{2^2}{3\cdot4}x^4+\dfrac{2^2\cdot4^2}{3\cdot4\cdot5\cdot6}x^6+\dfrac{2^2\cdot4^2\cdot6^2}{3\cdot4\cdot5\cdot6\cdot7\cdot8}x^8+\ldots.$[*Bundelkhand* **06**, *Bilaspur* **07**, *Garhwal* **09**, *Raipur* **11, 14**, *Allahabad* **14**]

5. $1+\dfrac{1}{2}x+\dfrac{1\cdot3}{2\cdot4}x^2+\dfrac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3+\ldots.$ [*Raipur* **2015**]

6. $x+\dfrac{2^2x^2}{2!}+\dfrac{3^3x^3}{3!}+\dfrac{4^4x^4}{4!}+\dfrac{5^5x^5}{5!}+\ldots.$

[*Himachal* **05**, *Indore* **06**, *Bilaspur* **06**, *Jabalpur* **07**, *Garhwal* **09**, *Avadh* **11**, *Allahabad* **13**]

ANSWERS

1. Con. if $a \le 0$, div. if $a > 0$. **2.** Convergent

3. Con. if $x^2 \le 1$, div. if $x^2 > 1$. **4.** Con. if $x^2 \le 1$, div. if $x^2 > 1$.

5. Con. if $x < 1$, div. if $x \ge 1$. **6.** Con. if $x < 1/e$, div. if $x \ge 1/e$.

6.7. Cauchy's Condensation Test.* *If the function $f(n)$ is positive for all positive integral values of n, and continually decreases as n increases; and if a is a positive integer greater than unity, then the two series,* $\sum f(n)$ *and* $\sum a^n f(a^n)$, *are both convergent or both divergent.*

Write the first series $\sum f(n)$ as follows:

$$\begin{aligned}&\{f(1) && +f(2) && +\dots+f(a)\}\\ &+\{f(a+1) && +f(a+2) && +\dots+f(a^2)\}\\ &+\{f(a^2+1) && +f(a^2+2) && +\dots+f(a^3)\}+\dots \quad (1)\end{aligned}$$

Then the terms in the rth group are

$$f(a^{r-1}+1)+f(a^{r-1}+2)+\dots+f(a^r). \qquad (2)$$

Each of these terms is greater than the last one, viz., $f(a^r)$, for the terms go on decreasing by hypothesis. Also the number of terms is $a^r - a^{r-1}$. Hence the sum of all the terms in (2) is greater than $(a^r - a^{r-1})f(a^r)$, that is

$$f(a^{r-1}+1)+f(a^{r-1}+2)+\dots$$
$$\dots+f(a^r)>(1-a^{-1})a^r f(a^r)$$

Giving to r the values 1, 2, 3, ..., n successively, and adding, we get

$$\sum_{2}^{a^n} f(r)>(1-a^{-1})\sum_{1}^{n} a^r f(a^r) \qquad (3)$$

Again, since the terms go on decreasing, each term in (2) is less than $f(a^{r-1})$, and their sum is less than $(a^r - a^{r-1})f(a^r)$, that is,

$$f(a^{r-1}+1)+f(a^{r-1}+2)+\dots$$
$$\dots+f(a^r)<(a-1)a^{r-1}f(a^{r-1})$$

Putting r equal to 1, 2, 3, .., n successively and adding, we get

$$\sum_{2}^{a^n} f(r)<(a-1)\sum_{1}^{n} a^{r-1}f(a^{r-1}) \qquad (4)$$

The inequality (4) shows that if $\sum a^r f(a^r)$ is convergent, so is $\sum f(r)$; while (3) shows that if $\sum a^r f(a^r)$ is divergent so also is $\sum f(r)$.

Generally it is immaterial what value is assigned to a.

6.71. The series $\sum 1/n(\log n)^p$. *The infinite series*

$$1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\dots+\frac{1}{n(\log n)^p}+\dots$$

is convergent if $p>1$, and divergent if $p \le 1$.

We shall apply Cauchy's condensation test to test the convergence of the given series. In the present case

$$f(n)=1/n(\log n)^p$$

and therefore

$$a^n f(a^n)=a^n/a^n(\log a^n)^p=1/(n\log a)^p$$
$$=(1/n^p)(\log a)^{-p}$$

The constant factor $(\log a)^{-p}$ is common to every term. Also, $\sum 1/n^p$ is convergent if $p>1$ and divergent if $p \le 1$. Therefore the given series is also convergent if $p>1$ and divergent if $p \le 1$.

6.72. Another Ratio Test. The above series can be used to devise the following test which can be applied when $\lim n(u_n/u_{n+1}-1)=1$ and so Raabe's test fails.

A series $\sum u_n$ of positive terms is convergent or divergent according as

$$\lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n>1 \text{ or } <1$$

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[*] Named after Augustin Louis Cauchy (1789-1857), a profound French mathematician. He was one of the leaders in infusing rigor in the treatment of infinite series.

Let $$\lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = l,$$

where $l > 1$, and choose a number p lying between l and 1. Then the series $\sum 1/n(\log n)^p$ is convergent, since $p > 1$. Comparing $\sum u_n$ with this series, we see that $\sum u_n$ will be convergent if

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p} \qquad (1)$$

for all sufficiently large values of n.

Now, for large values of n

$$\log(n+1) = \log n + \log(1+1/n) = \log n + 1/n + \dots,$$

and the right-hand side of (1) becomes

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\log n}+\dots\right)^p = \left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\log n}+\dots\right)$$

$= 1+\frac{1}{n}+\frac{p}{n\log n}+$ terms involving higher powers of n and $\log n$ in the denominator.

Hence the condition (1) can be written as

$$\frac{u_n}{u_{n+1}} > 1+\frac{1}{n}+\frac{p}{n\log n}+\dots,$$

i.e. $$n\left(\frac{u_n}{u_{n+1}}-1\right) > 1+\frac{p}{\log n}+\dots$$

or $$\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n > p+$$ terms involving n and $\log n$ in the denominator.

Now the left-hand side tends to l and the right-hand side tends to p as $n \to \infty$, and $l > p$. Therefore the inequality is satisfied for large values of n. Hence $\sum u_n$ is convergent.

The other case, when $l < 1$, can be proved similarly.

Ex. Test the convergence of the series

$$1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\dots.$$

[*Himachal* 06, *Jabalpur* 09]

Here $$u_n = \frac{2^2\cdot 4^2\dots(2n-2)^2}{3^2\cdot 5^2\dots(2n-1)^2}$$

and $$u_{n+1} = \frac{2^2\cdot 4^2\dots(2n-2)^2\,(2n)^2}{3^2\cdot 5^2\dots(2n-1)^2\,(2n+1)^2}$$

Therefore $$\frac{u_{n+1}}{u_n} = \frac{(2n)^2}{(2n+1)^2}.$$

which tends to 1 as $n \to \infty$. Proceeding to the next test, we see that

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\,\frac{(2n-1)^2-(2n)^2}{(2n)^2} = \frac{4n+1}{4n}.$$

which again tends to 1 as $n \to \infty$.

Finally, applying the test of the preceding article, we find that

$$\lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = \lim\left\{\frac{4n+1}{4n}-1\right\}\log n = \lim\frac{\log n}{4n} = 0$$

Since this limit is less than 1, the series is divergent.

This example is a particular case of the following general rule.

6.73. A general rule. *A series* $\sum u_n$ *of positive terms is divergent if*

$$n\left(\frac{u_n}{u_{n+1}}-1\right)$$

is an algebraic function of n which tends to 1 *as* $n \to \infty$.

Let $$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{n^k+a_1 n^{k-1}+\dots}{n^k+b_1 n^{k-1}+\dots}$$

the terms of the highest degree in the numerator and denominator being the same, because the limit of the expression is 1.

Then $$\lim \left\{ n\left(\frac{u_n}{u_{n+1}}-1\right)-1 \right\} \log n$$

$$= \lim \frac{(a_1-b_1)n^{k-1}+\dots}{n^k+b_1 n^{k-1}+\dots} \log n$$

$$= \lim \frac{(a_1-b_1)n^k+\dots}{n^k+\dots} \cdot \frac{\log n}{n}$$

Now the first factor tends to $a_1 - b_1$ and the second factor tends to zero as $n \to \infty$. Therefore the limit is zero, which is less than 1. Hence, by the theorem of the preceding article $\sum u_n$ is divergent.*

An independent proof can be given by comparing $\sum u_n$ directly with $\sum 1/n(\log n)^p$.

EXAMPLES

Find whether the following series are convergent or divergent

1. $\dfrac{(\log 2)^2}{2^2}+\dfrac{(\log 3)^2}{3^2}+\dots+\dfrac{(\log n)^2}{n^2}+\dots$ **[*Bundelkhand* 2011]**
2. $\dfrac{1}{(\log 2)^p}+\dfrac{1}{(\log 3)^p}+\dots+\dfrac{1}{(\log n)^p}+\dots$ [*Rohilkhand* 1995]
3. $\displaystyle\sum_{n=2}^{\infty}\frac{n^{-1}}{\sqrt{\log n}}$ [*Allahabad* 1995]
4. $\dfrac{a}{b}+\dfrac{a(a+1)}{b(b+1)}+\dfrac{a(a+1)(a+2)}{b(b+1)(b+2)}+\dots$ **[*Meerut* 2004]**
5. $1+\dfrac{a(1-a)}{1^2}+\dfrac{(1+a)a(1-a)(2-a)}{1^2.2^2}$

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* The theorem can be proved similarly if $n(u_n/u_{n+1}-1)$ involves fractional indices or the radical sign. By an algebraic function we mean that it does not involve n as a logarithm or exponent.

$$+\frac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2.2^2.3^2}+\dots$$ [*Rohilkhand* 1994]

Test the convergence of the following series for positive values of x.

6. $\dfrac{1^2}{2^2}+\dfrac{1^2.3^2}{2^2.4^2}x+\dfrac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\dots$ [*Rajasthan* 2003]
7. $1+\dfrac{\alpha.\beta}{1,\gamma}x+\dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\,\gamma(\gamma+1)}x^2$

$$+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma.(\gamma+1)(\gamma+2)}x^3+\dots,\quad x>0$$

[*Allahabad* 05, *Himachal* 06]

ANSWERS

1. Convergent
2. Divergent
3. Divergent
4. Con. if $b-a>1$, div. if $x \ge 1$.
5. Divergent
6. Con. if $x<1$, div. if $x \ge 1$.
7. Con. if $x<1$, div. if $x>1$; when $x=1$, con. if $\gamma-\alpha-\beta>0$, div. if $\gamma-\alpha-\beta \le 0$.

6.8. Test for Alternating series. *The infinite series* $u_1-u_2+u_3-\dots$, *in which the terms are alternately positive and negative, is convergent if each term is numerically less than the preceding term and* $\lim u_n = 0$.

Suppose $\quad u_1 > u_2 > u_3 > u_4 > \dots,$

and $\lim u_n = 0$. Consider the sum of an even number of terms. Then

$$s_{2n}=(u_1-u_2)+(u_3-u_4)+\dots+(u_{2n-1}-u_{2n}), \tag{1}$$

which can also be written as

$$s_{2n}=u_1-(u_2-u_3)-(u_4-u_5)-\dots-u_{2n}. \tag{2}$$

Since the terms are decreasing, the expressions within brackets in (1) and (2) are all positive. Therefore we see from (1) that s_{2n} is positive and increases as n increases; and we see from (2) that it always remains less than u_1. Hence s_{2n} must tend to a finite limit (§ 6.3, III).

Now consider the sum of an odd number of terms. Since

$$s_{2n+1} = s_{2n} + u_{2n+1}, \tag{3}$$

and $\lim u_{2n+1} = 0$, therefore s_{2n+1} tends to the same limit as s_{2n}.

Hence s_n tends to a finite limit whether n is even or odd. Therefore the series is convergent.

NOTE. The student should note that it is necessary for convergence that $\lim u_n = 0$. If $\lim u_n$ is not zero, we see from (3) that s_{2n+1} tends to a limit different form s_{2n}, and therefore the series is oscillatory. An example of such a series is $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots.$

If the terms are alternately positive and negative and go on increasing numerically, it can be shown that the series is oscillatory in this case also.

Ex. Show that the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$ is convergent.

[*Himachal* 2007]

Here the terms are alternately positive and negative, each term is numerically less than the preceding term, and $\lim u_n$, i.e. $\lim (1/n)$, is zero. Hence the series is convergent.

6.81. Changing the sign of terms. If we take a convergent series of positive terms and make the sign of some of the terms negative, the new series thus obtained is convergent.

For, if we denote by s_n and t_n respectively the sums of the first n terms of the two series then, numerically, t_n cannot be greater than s_n. Therefore t_n cannot tend to infinity. Also, $\lim u_n$ is zero. Therefore t_n cannot oscillate and must tend to a finite limit. Hence, the series is convergent.

6.82. Absolute Convergence. Consider the series $u_1 + u_2 + \ldots + u_n + \ldots$ in which any term may be either positive or negative. Let $|u_n|$ denote the absolute value of u_n, that is, $|u_n| = u_n$ if u_n is positive and $|u_n| = -u_n$ if u_n is negative. Then

$$|u_1| + |u_2| + \ldots + |u_n| + \ldots$$

is a series each term of which is positive and numerically equal to the corresponding term of $\sum u_n$.

If the series $\sum u_n$ is convergent, it is not necessary that $\sum |u_n|$ be also convergent. For example the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots.$$

is convergent, but the corresponding series of positive terms

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots.$$

is divergent. To distinguish between the two cases which can arise we give the following definitions :

A series $\sum u_n$ containing positive and negative terms, is said to be ***absolutely convergent*** *if the series $\sum |u_n|$ is convergent.*

$\sum u_n$ is said to be ***conditionally convergent*** *if $\sum u_n$ is convergent while $\sum |u_n|$ is divergent.*

Thus the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$ is conditionally convergent while the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

is absolutely convergent.

It will be found that in a conditionally convergent series, the positive terms and the negative terms taken separately form two divergent series. When their terms are suitably arranged, the difference gives a convergent series while their sum naturally gives a divergent series. In an absolutely convergent series, the positive and negative terms both form convergent series.

By §6.81, it follows that if $\sum |u_n|$ is convergent, $\sum u_n$ is also convergent. Therefore the absolute convergence of a series implies its ordinary convergence.

Ex. Discuss the convergence of the binomial series :

$$1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} x^r + \ldots \tag{1}$$

when n is not a positive integer.

We have $\dfrac{u_{r+1}}{u_r} = \dfrac{n-r+1}{r} x = -\left(1 - \dfrac{n+1}{r}\right) x.$ (2)

If x is positive, (2) is negative for large values of r and so the terms in (1) are alternately positive and negative ultimately. If x is negative, (2) is positive and ultimately the terms in (1) are all of the same sign. It is most convenient, therefore, to test the series for absolute convergence.

Now, by (2), $|u_{r+1}|/|u_r| \to |x|$ as $r \to \infty$.

Therefore the series is absolutely convergent if $|x| < 1$, i.e. if $-1 < x < 1$.

If $|x| > 1$, we see from (2) that $|u_{r+1}/u_r| > 1$ for all sufficiently large values of r. So the terms go on increasing numerically as r increases. Therefore the series is divergent (if the terms are ultimately of the same sign) or oscillatory (if the terms are alternately positive and negative).

When $x = -1$, the terms are ultimately of the same sign and $\lim u_{r+1}/u_r = 1$. Therefore, applying Raabe's test, we get

$$\lim r\left(\frac{u_r}{u_{n+1}} - 1\right) = \lim \frac{r(n+1)}{r-n-1} = n+1.$$

Hence the series is convergent for $n > 0$, and divergent for $n < 0$.

When $x = 1$, the terms are alternately positive and negative for large values of r, and we can apply the test of § 6.8. It can be shown that $\lim u_r = 0$ only when $n > -1$. Therefore the series is convergent for $n > -1$ and oscillatory for $n \leq -1$.

6.83. Steps for testing the convergence of a given series. When it is required to test a given series for convergence, it is convenient to apply the tests in the following order.

A glance will show whether the given series is (i) a series with terms alternately positive and negative; or (ii) a series of positive terms, which is not a power series *, or (iii) a power series of positive terms.

Step 1. For series (i) the test for alternating series (§ 6.8) should be applied.

For series (ii) an inspection should be made of the nth term, and if feasible $\lim u_n$ should be determined. If $\lim u_n \neq 0$, the series is divergent. If $\lim u_n = 0$, u_n may be comparable with $1/n^p$. If so, comparison test should be applied. If u_n involves $\log n$, Cauchy's condensation test can be applied.

Step 2. In case the above tests are inapplicable, the ratio test should be applied. If this fails on account of $\lim u_{n+1}/u_n$ being 1, Raabe's test (§ 6.61) should be applied. If this also fails for a similar reason, and u_n/u_{n+1} is an algebraic function of n, the series is divergent (§ 6.73).

Step 3. The logarithmic test (§ 6.6) should be applied as an alternative to Raabe's test when n occurs as an exponent in u_n/u_{n+1} The root test (§ 6.41) is an alternative to the ratio test and should only be applied in special cases in which the nth root can be easily evaluated.

Step 4. For the power series (iii) the ratio test should be applied first. If the series is in powers of x, it will be found that $\lim u_{n+1}/u_n = lx$ where l is some constant. For $x < 1/l$, the series will be convergent and for $x > 1/l$, divergent. For $x = 1/l$, the series should be treated just as in case (ii) above.

Step 5. If a power series is to be tested for such values of x which make the terms alternately positive and negative, the alternating series test (§ 6.8) should be applied.

EXAMPLES

Find whether the following series are convergent, divergent, or oscillatory :

1. $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots.$ [***Jabalpur*** **07,** ***Bundelkhand*** **09**]

2. $\dfrac{1}{x} - \dfrac{1}{x+a} + \dfrac{1}{x+2a} - \dfrac{1}{x+3a} + \ldots.$

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* A series in ascending powers of x (or some other number) is called a power series. Thus $a_0 + a_1 x + a_2 x^2 + \ldots\ldots,$ where $a_0, a_1, a_2, \ldots$ do not involve x, is a power series.

3. $\left(\frac{1}{2}-\frac{1}{\log 2}\right)-\left(\frac{1}{2}-\frac{1}{\log 3}\right)+\left(\frac{1}{2}-\frac{1}{\log 4}\right)-\left(\frac{1}{2}-\frac{1}{\log 5}\right)+\dots$

4. $\sum(-1)^n x^n / n.$

5. (i) $\sum\frac{(-1)^n n}{2n+3}$; (ii) $\sum\frac{(-1)^n x^n}{\sqrt{n}}$.

6. Are the following series absolutely convergent ?

(i) $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\dots,$

(ii) $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\dots.$ [*Indore* 2008]

(iii) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\dots,$

(iv) $1-2x+3x^2-4x^3+\dots.$

ANSWERS

1. Convergent 2. Convergent 3. Oscillatory

4. Convergent if $x \le 1$; oscillatory if $x > 1$.

5. (i) Oscillatory (ii) Convergent if $x \le 1$; oscillatory if $x > 1$.

6. (i) No (ii) Yes (iii) Yes (iv) Absolutely convergent if $x < 1$; not convergent if $x \ge 1$.

EXAMPLES (MISCELLANEOUS)

Find whether the following series is convergent or divergent.

1. $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\sqrt{\frac{4}{5}}+\dots.$

2. $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\dots.$

3. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots.$ [*Allahabad* 06, *Himachal* 07]

4. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\dots.$

5. $\frac{1}{\sqrt{2}-1}+\frac{1}{\sqrt{3}-1}+\frac{1}{\sqrt{4}-1}+\dots.$

6. $\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\dots+\frac{n}{1+n\sqrt{n+1}}+\dots.$

7. $1^p+\left(\frac{1}{2}\right)^p+\left(\frac{13}{24}\right)^p+\left(\frac{13.5}{24.6}\right)^p+\dots.$

[*Kashi Vidyapeeth* 08, *Allahabad* 12]

8. $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\dots.$

[*Allahabad* 14, *Raipur* 15]

Test for convergence the series whose nth term is:

9. $\frac{n^3+a}{2^n+a}$.

10. $\frac{\sqrt{n+1}-\sqrt{n}}{n}$.

11. $\sqrt{n^3+1}-\sqrt{n^3}$.

12. $(1-1/n)^{n^2}$.

13. $\cos(1/n)$

14. $\sin^2(1/n)$

15. $(\log n)/n.$

16. $1/n \log n \log\log n.$

Test the convergence of the following series for positive values of x:

17. $\sum\frac{a^n}{a^n+x^n}$.

18. $\sum\frac{1}{x^n+x^{-n}}$.

19. $\sum\frac{nx^n}{n^2+1}$.

20. $\sum\frac{n^{n^2}}{(1+n)^{n^2}}$.

21. $\sum\frac{3n+1}{4n+3}x^n$.

22. $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\dots+\frac{(n+1)x^n}{n^3}+\dots.$

[*Garhwal* 08, 10, *Kashi Vidyapeeth* 12]

23. $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\dots.$

[*Rohilkhand* 07, *Indore* 07, *Jabalpur* 05, 08, *Raipur* 13]

24. $1+\frac{1}{2}x+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\dots.$

[*Bilaspur* 08, *Raipur* 12, 14]

25. $\frac{a+x}{1}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\dots.$

26. $x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \ldots$

27. $x + x^{1+1/2} + x^{1+1/2+1/3} + \ldots$ [*Allahabad* 11, 14]

28. If $\dfrac{u_n}{u_{n+1}} = \dfrac{n^k + An^{k-1} + Bn^{k-2} + Cn^{k-3} + \ldots}{n^k + an^{k-1} + bn^{k-2} + cn^{k-3} + \ldots}$

where k is a positive integer, show that the series $\sum u_n$ is convergent if $A - a - 1 > 0$ and divergent if $A - a - 1 \le 0$.

29. If $\sum u_n$ is a convergent series of positive terms, prove that $\sum u_n^2$ will also be convergent

30. Test the convergence of $\sum \dfrac{U_n}{1 - U_n}$ if $\sum U_n$ is convergent and $U_n \neq 1$.

ANSWERS

1. Divergent. **2.** Convergent if $p > 1$; divergent if $p \le 1$.
3. Convergent. **4.** Convergent. **5.** Divergent.
6. Divergent. **7.** Convergent if $p > 2$; divergent if $p \le 2$.
8. Convergent. **9.** Convergent. **10.** Convergent.
11. Convergent. **12.** Convergent. **13.** Divergent.
14. Divergent. **15.** Divergent. **16.** Divergent.
17. Divergent if $x \le a$, convergent if $x > a$, supposing $a > 0$.
18. Convergent if $x \neq 1$. **19.** Convergent if $x < 1$; divergent if $x \ge 1$.
20. Convergent. **21.** Convergent if $x < 1$; divergent if $x \ge 1$.
22. Convergent if $x \le 1$; divergent if $x > 1$.
23. Convergent. if $x \le 1$; divergent if $x > 1$.
24. Convergent if $x < e$; divergent if $x \ge e$.
25. Convergent if $x < 1/e$; divergent if $x \ge 1/e$.
26. Convergent if $x < 1$; divergent if $x \ge 1$; the result holds for all values of q, positive, zero or negative.
27. Convergent if $x < 1/e$; divergent if $x \ge 1/e$.
30. Convergent.

7

Expansion of Functions

7.1. Introduction. We have seen that the ordinary processes of addition, subtraction, multiplication, division, rearrangement of terms, raising to a given power, taking limits, differentiation, etc., though applicable to the sum of a finite number of terms, may break down for an infinite series. The expansions in the form of infinite series obtained by the methods given below are therefore to be regarded merely as formal expansions, which may not be true in exceptional cases.

7.11. Maclaurin's Theorem. [*Bundelkhand* 06, *Avadh* 05, 08, *Bilaspur* 07, *Raipur* 11, *Kashi Vidyapeeth* 12]

Let $f(x)$ be a function of x which can be expanded in powers of x and let the expansion be differentiable term by term any number of times.

Suppose $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$.

Then by successive differentiation we have

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \ldots,$$

$$f''(x) = 2.1\,A_2 + 3.2\,A_3 x + 4.3\,A_4 x^2 + \ldots,$$

$$f'''(x) = 3.2.1 A_3 + 4.3.2 A_4 x + \ldots; \text{ etc.}$$

Putting $x = 0$ in each of these, we get

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = 2!\,A_2,\ f'''(0) = 3!\,A_3, \ldots$$

Hence $\mathbf{f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!} f''(0) + \dfrac{x^3}{3!} f'''(0) + \ldots + \dfrac{x^n}{n!} f^n(0) + \ldots}$

This result is generally known as **Maclaurin's Theorem.**

Ex. 1. Expand $\sin x$ by Maclaurin's theorem. [*Delhi* 06, *Patna* 08]

Let $f(x) = \sin x$ then $f(0) = 0,$

$$f'(x) = \cos x, \qquad f'(0) = 1,$$

$$f''(x) = -\sin x, \qquad f''(0) = 0,$$

$$f'''(x) = -\cos x, \qquad f'''(0) = -1,$$

etc. etc.

$$f^{(n)}(x) = \sin(x + \tfrac{1}{2}n\pi), \qquad f^{(n)}(0) = \sin \tfrac{1}{2}n\pi$$

$$= 0 \text{ if } n = 2m,$$

and $= (-1)^m$ if $n = 2m + 1.$

Hence

$$\sin x = 0 + x.1 + 0 + \frac{x^3}{3!}(-1) + 0 + \ldots + 0 + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots.$$

Ex. 2. Apply Maclaurin's theorem to prove that

$$\log \sec x = \tfrac{1}{2}x^2 + \tfrac{1}{12}x^4 + \tfrac{1}{45}x^6 + \ldots.$$

[*Jabalpur* 04, *Indore* 07, *Agra* 09, *Lucknow* 10, *Bundelkhand* 13]

Let $y = \log \sec x$ then $y(0) = \log 1 = 0$

$$y_1 = \frac{\sec x \tan x}{\sec x} = \tan x \qquad y_1(0) = \tan 0 = 0 \tag{1}$$

$$y_2 = \sec^2 x \qquad y_2(0) = \sec^2 0 = 1 \tag{2}$$

$$y_3 = 2\tan x \sec^2 x = 2y_1 y_2 \qquad y_3(0) = y_1(0)y_2(0) = 0 \tag{3}$$

Using (1), (2) and (3), we get

$$y_4 = 2y_2 y_2 + 2y_1 y_3 = 2(y_2^2 + y_1 y_3) y_4(0) = 2(1+0) = 2 \tag{4}$$

Using (1), (2), (3) and (4), we get

$y_5 = 2(2y_2 y_3 + y_2 y_3 + y_1 y_4) = 2(y_1 y_4 + 3y_2 y_3)$ so that

$y_5(0) = 2(0+0) = 0.$

Again, $y_6 = 2(y_2 y_4 + y_1 y_5 + 3y_3^2 + 3y_2 y_4) = 2(y_1 y_5 + 4y_2 y_4 + 3y_3^2)$

so that $y_6(0) = 2(0 + 4.1.2 + 0) = 16.$

Hence by Maclaurin's theorem,

$$\log \sec x = 0 + x.0 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.2 + \frac{x^5}{5!}.0 + \frac{x^6}{6!}.16 + \ldots\ldots$$

or $$\log \sec x = \tfrac{1}{2}x^2 + \tfrac{1}{12}x^4 + \tfrac{1}{45}x^6 + \ldots.$$

Ex. 3. Apply Maclaurin's theorem to show that

$$e^x \sin x = x + x^2 + \frac{2x^3}{3!} - \frac{2^2 x^5}{5!} - \ldots + 2^{n/2} \sin \tfrac{1}{4}n\pi \cdot \frac{x^n}{n!} + \ldots.$$

[*Gorakhpur* 06, *Lucknow* 08, 11, *Kumaon* 11]

Let $y = e^x \sin x,$ then $y(0) = 0.$

The formula for the nth derivative is

$$D^n\{e^{ax}\sin(bx + c)\} = r^n e^{ax} \sin(bx + c + n\phi) \tag{1}$$

where $r = (a^2 + b^2)^{1/2}$ and $\phi = \tan^{-1}(b/a).$

Taking $a = b = 1$ and $c = 0$ in (1), we have

$$y_n = D^n(e^x \sin x) = 2^{n/2} e^x \sin(x + \tfrac{1}{4}n\pi) \tag{2}$$

since $r = 2^{1/2}$ and $\phi = \tfrac{1}{4}\pi.$

Now taking $n = 1, 2, 3, 4, \ldots\ldots$ and putting $x = 0$, we get from

$$y_1 = 2^{1/2} e^x \sin(x + \tfrac{1}{4}\pi) \qquad y_1(0) = 2^{1/2}.\sin \tfrac{1}{4}\pi = 1.$$

$$y_2 = 2 \ e^x \sin(x + \tfrac{1}{2}\pi) \qquad y_2(0) = 2 \cdot \sin \tfrac{1}{2}\pi = 2.$$

$$y_3 = 2^{3/2} e^x \sin(x + \tfrac{3}{4}\pi) \qquad y_3(0) = 2^{3/2}.\sin \tfrac{3}{4}\pi = 2^{3/2} \cdot \tfrac{1}{\sqrt{2}} = 2.$$

$$y_4 = 2^2 \ e^x \sin(x + \ \pi) \qquad y_4(0) = 2^2.\sin \pi = 2^2.0 = 0.$$

$$y_5 = 2^{5/2} e^x \sin(x + \tfrac{5}{4}\pi) \qquad y_5(0) = 2^{5/2}.\sin \tfrac{5}{4}\pi = 2^{5/2}(-\tfrac{1}{\sqrt{2}}) = -2^2.$$

Hence by Maclaurin's theorem,

$$e^x \sin x = 0 + 1.x + 2.\frac{x^2}{2!}x^2 + \frac{2.x^3}{3!} - \frac{2^2 x^5}{5!} - \ldots + 2^{n/2} \sin \tfrac{1}{4}n\pi \cdot \frac{x^n}{n!} + \ldots.$$

or $$e^x \sin x = x + x^2 + \frac{2x^3}{3!} - \frac{2^2 x^5}{5!} - \ldots + 2^{n/2} \sin \tfrac{1}{4}n\pi \cdot \frac{x^n}{n!} + \ldots.$$

Ex. 4. With the help of Maclaurin's theorem give the expansion of $f(x) = \tan x$ in ascending powers of x. [*Jabalpur* 2003]

Let $y = \tan x$ then $y(0) = 0$

$$y_1 = \sec^2 x \qquad y_1(0) = 1$$

$$y_2 = 2\sec^2 x \tan x = 2yy_1 \qquad y_2(0) = 0$$

$$y_3 = 2y_1y_1 + 2yy_2$$

$$= 2{y_1}^2 + 2yy_2 \qquad y_3(0) = 2$$

$$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3$$

$$= 6y_1y_2 + 2yy_3 \qquad y_4(0) = 0$$

$$y_5 = 6y_2y_2 + 6y_1y_3 + 2y_1y_3 + 2yy_4$$

$$= 6{y_2}^2 + 8y_1y_3 + 2yy_4 \qquad y_5(0) = 16$$

Using Maclaurin's theorem, we get

$$\tan x = 0 + x \,.\, 1 + \frac{x^2}{2!} \,.\, 0 + \frac{x^3}{3!} \,.\, 2 + \frac{x^4}{4!} \,.\, 0 + \frac{x^5}{5!} \,.\, 16 + \ldots\ldots$$

or $$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\ldots$$

NOTE. When the *n*th differential coefficient of the function cannot be found, the *n*th term of the expansion cannot be ascertained. It is possible, however, that the *n*th differential coefficient be known for $x = 0$.

EXAMPLES

Expand the following functions by Maclaurin's theorem:

1. $\cos x$. [*Kanpur* 2006] ② $\sec x$. [*Bilaspur* 2009]

3. $\log(1+\sin^2 x)$. 4. $\sin^{-1} x$. [*Agra* 2010]

5. $\tan^{-1} x$. [*Indore* 06, *Avadh* 06, *Jabalpur* 08, 09]

6. $\log(1+x)$. [*Himachal* 05, *Garhwal* 07, *Jabalpur* 09,

[*Agra* 05, 10, *Meerut* 11]

7. $e^x \sec x$. or $\dfrac{e^x}{\cos x}$. [*Meerut* 04, *Agra* 04]

Apply Maclaurin's theorem to show that

⑧ $e^{\sin x} = 1 + x + \dfrac{x^2}{2!} - \dfrac{3x^4}{4!} + \ldots$ [*Bundelkhand* 2007]

⑨ $e^x \cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{2^2x^4}{4!} - \dfrac{2^2x^5}{5!} + \dfrac{2^3x^7}{7!} + \ldots$

[*Raipur* 12, *Bundelkhand* 14]

10. Prove that. [*Jabalpur* 2008]

$$\log(1+\sin x) = x - x^2/2! + x^3/3! - 2x^4/4! + 5x^5/5! + \ldots$$

OR Obtain by Maclaurin's theorem the first five terms in the expansion of $\log(1+\sin x)$. [*Meerut* 2007]

11. Prove that $\log\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - \ldots\ldots$

[*Jabalpur* 2007]

12. Prove that

$$e^{ax}\cos bx = 1 + ax + \frac{a^2-b^2}{2!}x^2 + \frac{a(a^2-3b^2)}{3!}x^3 + \ldots$$

$$\ldots + \frac{(a^2-b^2)^{n/2}}{n!}x^n \cos\{n\tan^{-1}(b/a)\} + \ldots$$

[*Kumaon* 2012]

13. Expand $e^{ax}\cos bx$ by Maclaurin's theorem and deduce that $e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \dfrac{x^2}{2!}\cos 2\alpha + \dfrac{x^3}{3!}\cos 3\alpha + \ldots$

[*Rohilkhand* 07, *Avadh* 11]

14. Expand $\log\{1-\log(1-x)\}$ in powers of x by Maclaurin's theorem as far as the term in x^3. [*Avadh* 2009]

15. Expand $\log_e(1+e^x)$ in powers of x.

[*Kanpur* 11, *Rohilkhand* 12]

16. Find the first three terms in the expansion of $e^x/(e^x+1)$ in powers of x by Maclaurin's theorem [*Lucknow* 2006]

17. Expand $e^{a\sin^{-1}x}$ in powers of x by Maclaurin's theorem and find the general term.

Hence show that $e^{\theta} = 1 + \sin\theta + \dfrac{\sin^2\theta}{2!} + 2\dfrac{\sin^3\theta}{3!} + \ldots$

[*Rewa* 05, *Jabalpur* 05, *Kumaon* 08, *Lucknow* 12]

ANSWERS

1. $1 - x^2/2! + x^4/4! - x^6/6! + \ldots + (-1)^n x^{2n}/(2n)! + \ldots$
2. $1 + x^2/2! + 5x^4/4! + 61x^6/6! + \ldots$
3. $x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 + \ldots$
4. $x + 1^2.x^3/3! + 3^2.1^2.x^5/5! + 5^2.3^2.1^2.x^7/7! + \ldots$
5. $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots + (-1)^{n-1}x^{2n-1}/(2n-1) + \ldots$
6. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots + (-1)^{n-1}x^n/n + \ldots$
7. $1 + x + x^2 + \frac{2}{3}x^3 + \ldots$

14. $x+\frac{1}{6}x^3+\ldots.$

15. $\log_e 2+\frac{1}{2}x+\frac{1}{4}x^2/2!-\frac{1}{8}x^4/4!+\ldots.$

16. $\frac{1}{2}+\frac{1}{4}x-\frac{1}{8}x^3/3!+\ldots.$

17. $1+ax+\frac{x^2}{2!}a^2+\frac{x^3}{3!}(1+a^2)a+\frac{x^4}{4!}(2^2+a^2)a^2+\ldots.$

7.2. Taylor's Theorem. Let $f(a+h)$ be a function of h which can be expanded in powers of h, and let the expansion be differentiable any number of times with respect to h.

Suppose $f(a+h)=A_0+A_1h+A_2h^2+A_3h^3+\ldots.$

By successive differentiation with respect to h, we have

$$f'(a+h)=A_1+2A_2h+3A_3h^2+4A_4h^3+\ldots.$$

since $\frac{d}{dh}f(a+h)=\frac{d}{dh}f(t).\frac{dt}{dh}$, where $t=a+h$,

$$=f'(t).1=f'(a+h);$$

$$f''(a+h)=2.1A_2+3.2A_3h+4.3A_4h^2+\ldots.$$

$$f'''(a+h)=3.2.1.A_3+4.3.2A_4h+\ldots. \quad \text{etc.}$$

Putting $h=0$ in each of these, we have

$$f(a)=A_0, f'(a)=A_1, f''(a)=2!A_2, f'''(a)=3!A_3, \ldots.$$

Hence

$$\boxed{f(a+h)=f(a)+\underset{1!}{h}f'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots..+\frac{h^n}{n!}f^n(a)+\ldots.}$$

This is known as **Taylor's Theorem.**

→*If we put $a=0$, and $h=x$ in the above result, we get the particular case known as Maclaurin's Theorem.*

Taylor's theorem is generally quoted as

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots+\frac{h^n}{n!}f^n(x)+\ldots$$

which is obtained at once from the above on replacing a by x.

A more useful form is obtained on replacing h by $(x-a)$.

Thus $$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots+\frac{(x-a)^n}{n!}f^n(a)+\ldots$$

which is an expansion in powers of $(x-a)$.

Ex. 1. Expand $\log\sin x$ in powers of $(x-2)$.

[*Poorvanchal* 04, *Agra* 04, *Allahabad* 05, *Avadh* 07, *Raipur* 10, 14]

Let $f(x)=\log\sin x$. (1)

This can be written as $f(2+x-2)=\log\sin(\underset{a}{2}+\underset{h}{x-2})$

since (1) is to be expanded in powers of $(x-2)$.

Then $f(2)=\log\sin 2$

$f'(x)=\cot x, \qquad f'(2)=\cot 2$

$f''(x)=-\text{cosec}^2 x, \qquad f''(2)=-\text{cosec}^2 2$

$f'''(x)=2\text{cosec}^2 x.\cot x, \quad f'''(2)=2\text{cosec}^2 2\cot 2$, etc.

Hence by Taylor's theorem, we have

$$f(x)=f(2+x-2)=f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+\ldots.$$

or $$\log\sin x=\log\sin 2+(x-2)\cot 2+\frac{(x-2)^2}{2!}(-\text{cosec}^2 2)+\frac{(x-2)^3}{3!}(2\text{cosec}^2 2\cot 2)+\ldots.$$

Ex. 2. Expand $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$.

[*Meerut* 05, *Agra* 04, 06, *Jabalpur* 06, *Lucknow* 07, *Bundelkhand* 07, 08, *Avadh* 08, *Rohilkhand* 06, 10, *Kashi Vidyapeeth* 12]

Let $f(x)=\sin x$ then $f(\frac{1}{2}\pi)=1$

$f'(x)=\cos x \qquad f'(\frac{1}{2}\pi)=0$

$f''(x)=-\sin x \qquad f''(\frac{1}{2}\pi)=-1$

$$f'''(x) = -\cos x \qquad f'''\left(\tfrac{1}{2}\pi\right) = 0$$

$$f^{iv}(x) = \sin x \qquad f^{iv}\left(\tfrac{1}{2}\pi\right) = 1 \text{ etc.}$$

By applying Taylor's theorem, we get

$$\sin x = f(\pi/2) + (x-\pi/2)f'(\pi/2) + \frac{1}{2!}(x-\pi/2)^2 f''(\pi/2)$$

$$+\frac{1}{3!}(x-\pi/2)^3 f'''(\pi/2) + \frac{1}{4!}(x-\pi/2)^4 f^{iv}(\pi/2) + \ldots$$

or $\sin x = 1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 - \ldots$.

Ex. 3. Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

[***Indore* 05, *Gorakhpur* 06, *Raipur* 09, *Allahabad* 10, *Kashi Vidyapeeth* 10, *Agra* 11, *Kanpur* 11, *Lucknow* 12**]

Let $f(x) = f(2 + x - 2) = 2x^3 + 7x^2 + x - 1$

Then $f(2) = 22^3 + 7.2^2 + 2 - 1 = 45$

$$f'(x) = 6x^2 + 14x + 1, \qquad f'(2) = 53$$

$$f''(x) = 12x + 14, \qquad f''(2) = 38$$

$$f'''(x) = 12, \qquad f'''(2) = 12$$

and $f^{iv}, f^{v}, \ldots$ are all zero. Hence by Taylor's theorem,

$$f(x) = f(2) + (x-2)f'(2) + \frac{1}{2!}(x-2)^2 f''(2)$$

$$+\frac{1}{3!}(x-2)^3 f'''(2) + \ldots$$

$$= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$$

Ex. 4. Prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin z\,\frac{\sin z}{1} - (h\sin z)^2\,\frac{\sin 2z}{2}$$

$$+(h\sin z)^3\,\frac{\sin 3z}{3} + \ldots, \qquad \text{where } z = \cot^{-1}x.$$

[***Rewa* 05, *Gorakhpur* 05, *Agra* 07, *Bilaspur* 10, *Kashi Vidyapeeth* 11, *Raipur* 11, *Kumaon* 13**]

Let $f(x) = \tan^{-1}x$, then $f(x+h) = \tan^{-1}(x+h)$

Since $z = \cot^{-1}x$, therefore

$$\frac{dz}{dx} = -\frac{1}{1+x^2} = -\frac{1}{1+\cot^2 z} = -\frac{1}{\operatorname{cosec}^2 z} = -\sin^2 z.$$

Now, $f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \frac{1}{\operatorname{cosec}^2 z} = \sin^2 z.$

$$f''(x) = \frac{d}{dx}(\sin^2 z) = 2\sin z\cos z\,\frac{dz}{dx}$$

$$= -2\sin z\cos z\sin^2 z \qquad [\because dz/dx = -\sin^2 z]$$

$$= -\sin 2z\sin^2 z$$

$$f'''(x) = \frac{d}{dx}[-\sin 2z\sin^2 z] = -\frac{d}{dz}[\sin 2z\sin^2 z]\frac{dz}{dx}$$

$$= -[2\cos 2z\sin^2 z + \sin 2z\,2\sin z\cos z](-\sin^2 z)$$

$$= 2\sin^3 z\,[\cos 2z\sin z + \sin 2z\cos z]$$

$$= 2\sin^3 z\sin(z+2z) = 2\sin^3 z\sin 3z \text{ and so on}$$

Now, by Taylor's Theorem

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

So $\tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1!}\sin z\sin z - \frac{h^2}{2!}(-\sin 2z\sin^2 z)$

$$+\frac{h^3}{3!}2\sin^3 z\sin 3z + \ldots$$

$$\therefore \quad \tan^{-1}(x+h) = \tan^{-1}x + (h\sin z)\frac{\sin z}{1} - (h\sin z)^2\cdot\frac{\sin 2z}{2}$$

$$+(h\sin z)^3\cdot\frac{\sin 3z}{3} - \ldots.$$

7.3. Expansion by Algebraic and Trigonometrical Methods. When we want only the first few terms of an expansion, it is very often more convenient to use the Binomial, Exponential or the Logarithmic theorems, or the well-known expansions of sin x and cos x, in conjunction with algebraic or trigonometrical methods.

But care should be taken not to omit inadvertently any term of an order lower than, or the same as, that of the highest order term retained.

Ex. Expand $e^{\sin x}$ as far as the term containing x^4.

We have $e^{\sin x} = e^{(x - x^3/6 + \ldots)}$

$$= 1 + \left(x - \tfrac{1}{6}x^3 + \ldots\right) + \tfrac{1}{2}\left(x - \tfrac{1}{2}x^3 + \ldots\right)^2$$

$$+\tfrac{1}{6}\left(x - \tfrac{1}{6}x^3 + \ldots\right)^3 + \tfrac{1}{24}(x - \ldots)^4 + \ldots$$

$$=1+x-\tfrac{1}{6}x^3+\ldots+\tfrac{1}{2}(x^2-\tfrac{1}{3}x^4+\ldots)+\tfrac{1}{6}(x^3+\ldots)+\tfrac{1}{24}(x^4+\ldots)+\ldots$$

$$=1+x+\tfrac{1}{2}x^2-\tfrac{1}{8}x^4+\ldots.$$

7.32. By differentiation of a known series. Occasionally it is possible to derive an expansion by differentiating a known series.

Ex. Show that $\sec^2 x=1+x^2+\tfrac{2}{3}x^4+\ldots.$ [*Lucknow* 2004]

We know that $\tan x=x+\tfrac{1}{3}x^3+\tfrac{2}{15}x^5+\ldots.$

Hence, by differentiation, $\sec^2 x=1+x^2+\tfrac{2}{3}x^4+\ldots.$

EXAMPLES

1. Using Taylor's theorem, prove that

$$\log(x+h)=\log h+x/h-x^2/2h^2+x^3/3h^3-\ldots.$$

[*Jabalpur* 08, 09]

Expand the following in powers of the quantity indicated.

2. $\cos x$ in powers of $x-\tfrac{1}{4}\pi$.
3. $\log\sin(x+h)$ in powers of h by Taylor's theorem. [*Bundelkhand* 09, *Meerut* 10]
4. $\log_e\cos(x+h)$ in powers of h.
5. $\tan^{-1}x$ in powers of $(x-\tfrac{1}{4}\pi)$ by Taylor's theorem. [*Bundelkhand* 12, *Raipur* 15]
6. $\sin(\tfrac{1}{4}\pi+\theta)$ in powers of θ. [*Lucknow* 09, 11]
7. Prove that $\log(x+h)=\log h+\dfrac{x}{h}-\dfrac{x^2}{2h^2}+\dfrac{x^3}{3h^3}-\ldots.$ [*Rewa* 04, *Jabalpur* 08, 09]
8. Prove that $\dfrac{1}{x+h}=\dfrac{1}{x}-\dfrac{h}{x^2}+\dfrac{h^2}{x^3}-\dfrac{h^3}{x^4}+\ldots..$ [*Bilaspur* 2008]
9. Prove that

$$f(mx)=f(x)+(m-1)x.f'(x)+\frac{1}{2!}(m-1)^2x^2.f''(x)+\frac{1}{3!}(m-1)^3x^3.f'''(x)+\ldots..$$

[*Agra* 2007]

[HINT: Write $f(mx)$ as $f\{x+(m-1)x\}$ and apply Taylor's Theorem.]

10. Prove that $f\left(\dfrac{x^2}{1+x}\right)=f(x)-\left(\dfrac{x}{1+x}\right)f'(x)+\left(\dfrac{x}{1+x}\right)^2\dfrac{f''(x)}{2!}-\ldots.$

[*Rohilkhand* 08, *Raipur* 13]

[HINT: Write $f\left(\dfrac{x^2}{1+x}\right)$ as $f\left(x-\dfrac{x}{1+x}\right)$ and apply Taylor's Theorem.]

ANSWERS

2. $\dfrac{1}{\sqrt{2}}\left[1-(x-\tfrac{1}{4}\pi)-\dfrac{(x-\tfrac{1}{4}\pi)^2}{2!}+\dfrac{(x-\tfrac{1}{4}\pi)^3}{3!}+\ldots\right].$

3. $\log\sin x+h\cot x-\tfrac{1}{2}h^2\operatorname{cosec}^2x+\tfrac{1}{3}h^3\operatorname{cosec}^2x\cot x-\ldots\ldots$

4. $\log_e\cos x-h\tan x-\tfrac{1}{2}h^2\sec^2x-\tfrac{1}{3}h^3\sec^2x\tan x-\ldots$

5. $\tan^{-1}\tfrac{1}{4}\pi+(x-\tfrac{1}{4}\pi)/(1+\tfrac{1}{16}\pi^2)-\pi(x-\tfrac{1}{4}\pi)^2/4(1+\tfrac{1}{16}\pi^2)^2+\ldots$

6. $(1/\sqrt{2})[1+\theta-\theta^2/2!-\theta^3/3!+\theta^4/4!+\theta^5/5!-\ldots]$

7.4. Rolle's Theorem. *If $f(x)$ is a function, which is*

(i) *continuous in the closed interval* $[a, b]$,

(ii) *differentiable in the open interval* (a, b),

and (iii) $f(a)=f(b)$,

then there exists at least one value of x, *say* $c\in(a, b)$ *such that* $f'(c)=0$.

[*Meerut* 05, *Himachal* 05, 07, *Avadh* 08, *Lucknow* 08, *Garhwal* 06, 09, *Jabalpur* 09, *Kanpur* 08, 12, *Bundelkhand* 12, *Raipur* 14]

To prove Rolle's theorem, consider the following cases :

Case 1. The truth of this theorem is obvious if $f(x)$ is a constant. Then $f'(x)=0$ for all $x\in(a, b)$. Geometrically, $f(x)$ is a straight line parallel to the x-axis.

Case 2. If $f(x)$ is not a constant, then since it is continuous in the closed interval $[a, b]$, it will either increase or decrease as x takes values greater than a.

Suppose $f(x)$ increases for $x>a$. Since $f(a)=f(b)$, the function will continue increasing till it reaches a maximum point, say $c\in(a, b)$ and then begin to decrease.

Obviously, $c \neq a$ and $c \neq b$. Now as $f(x)$ has a maximum value at $x = c$, then for any very small positive number h,

$$f(c+h) - f(c) \leq 0 \quad \text{and} \quad f(c-h) - f(c) \leq 0$$

or
$$\frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{and} \quad \frac{f(c-h) - f(c)}{-h} \geq 0.$$

Taking limits as $h \to 0$, we have

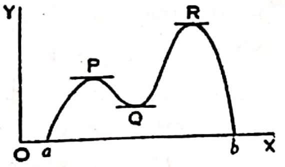
$$R\,f'(c) \leq 0 \qquad \text{and} \qquad L\,f'(c) \geq 0.$$

But since $f(x)$ is differentiable in (a, b), the only possibility for $R\,f'(c) = L\,f'(c)$ is $f'(c) = 0$.

Case 3. The proof of this theorem is similar when $f(x)$ decreases for values of x greater than a.

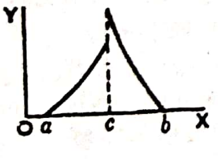
Geometrical Meaning of Rolle's theorem. The theorem can be geometrically interpreted as follows :

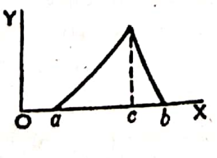
If the ordinates at the end points A and B are equal, then there exists at least one point on the curve between these ordinates where the *tangent is parallel to the x-axis.* There might be more than one such point (as in the figure).



If $f(x)$ is discontinuous at any point between a and b, or if $f'(x)$ does not exist at some point, the theorem may not be true.

Thus, in the figures given below, $f'(x)$ does not exist at $x = c$. In these cases there is no point between a and b where the tangent to the curve is parallel to the x-axis.





Ex. 1. Verify Rolle's theorem for the function

$$f(x) = \sin x + \cos x - 1 \qquad \text{in} \qquad [0, \pi/2].$$

Since the sine, cosine and the constant functions are continuous, it follows that $f(x)$ is continuous in the closed interval $[0, \pi/2]$.

Also $f'(x) = \cos x - \sin x$ exists for all $x \in [0, \pi/2]$, so $f(x)$ is differentiable in the open interval $(0, \pi/2)$.

We have $f(0) = f(\pi/2) = 0$.

Since all the conditions of Rolle's theorem are satisfied, hence there must exist a $c \in (0, \pi/2)$ such that $f'(c) = 0$.

i.e. $\cos c - \sin c = 0$ or $\cos c = \sin c$ or $c = \pi/4$ which lies in the interval $(0, \pi/2)$. Hence Rolle's theorem is verified.

Ex. 2. Verify Rolle's theorem for the function

$$f(x) = (x-a)^2 (x-b)^3 \text{ for all } x \in [a, b]$$ [***Rohilkhand* 2012**]

On expanding the polynomials $(x-a)^2$ and $(x-b)^3$ by the Binomial theorem, their product $f(x)$ is again a polynomial of the 5th degree in x. Hence f is both continuous and differentiable for all real values of x.

Also
$$f(a) = f(b) = 0.$$

Thus $f(x)$ satisfies all the conditions of Rolle's theorem, hence there must exist a $c \in (a, b)$ such that $f'(c) = 0$.

Now
$$\begin{aligned} f'(x) &= 2(x-a)(x-b)^3 + 3(x-a)^2(x-b)^2 \\ &= (x-a)(x-b)^2[2(x-b) + 3(x-a)] \\ &= (x-a)(x-b)^2(5x - 3a - 2b). \end{aligned}$$

So $f'(c) = 0$ gives $(c-a)(c-b)^2(5c - 3a - 2b) = 0$

i.e.
$$c = a, b \qquad \text{or} \qquad \tfrac{1}{5}(3a + 2b).$$

Since c lies in the open interval (a, b), hence $c \neq a$ and $c \neq b$.

Thus $c = \frac{1}{5}(3a + 2b)$ which lies in (a, b) verifying the theorem.

Ex. 3. Verify Rolle's theorem for the function $f(x) = |x|$ in $[-1, 1]$.

We have $f(1) = |1| = 1$ and $f(-1) = |-1| = 1$.

i.e.
$$f(1) = f(-1) = 1$$

We can see that $f(x)$ is continuous for all values of x in $[-1, 1]$ except maybe at $x = 0$.

Here $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} |0+h| = 0$

and $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} |0-h| = 0$

Also $f(0) = |0| = 0$

Therefore $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0)$

Hence $f(x)$ is continuous at $x=0$ and thus at every point in $[-1, 1]$.

To test if $f(x)$ is derivable at $x=0$, we see that

$$f'(0+0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \text{ and}$$

$$f'(0-0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|}{h} = \lim_{h\to 0} \frac{h}{-h} = -1.$$

Since $f'(0+0) \neq f'(0-0)$, the function is not differentiable at $x=0$ and hence the condition of differentiability at each point in the open interval $(-1, 1)$ is not satisfied. Therefore, Rolle's theorem is not applicable to $f(x) = |x|$ in $[-1, 1]$.

Ex. 4. Verify whether Rolle's theorem is applicable to the function $f(x) = 2 + (x-1)^{2/3}$ in the interval $[0, 2]$ or not. **[*Lucknow* 2006]**

If a function fails to satisfy even one condition of Rolle's theorem, then the theorem will not be applicable to the function. Let us consider the differentiability of $f(x)$ in the open interval $(0, 2)$.

Now $f'(x) = \dfrac{2}{3(x-1)^{1/3}}$ which does not exist at $x=1$.

So $f(x)$ is not differentiable at $x=1$ and therefore in $(0, 2)$. Hence Rolle's theorem is not applicable to the given function.

EXAMPLES

Verify Rolle's theorem for the following functions:

1. $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$. **[*Jabalpur* 07, *Bilaspur* 08]**
2. $f(x) = x(x+3)e^{-x/2}$, in $[-3, 0]$. **[*Kumaon* 2011]**
3. $f(x) = 10x - x^2$ in $[0, 10]$. **[*Kanpur* 2006]**
4. $f(x) = x^2$ in $[-1, 1]$. **[*Lucknow* 2010]**
5. $f(x) = (x-1)^2(x+1)^3$ in $[-1, 1]$. **[*Bundelkhand* 2014]**
6. $f(x) = (x^2 + 2x - 3)e^x$ in $[-3, 1]$. **[*Kanpur* 2004]**

8. $f(x) = \sin x$ in $[0, \pi]$. **[*Bundelkhand* 2009]**

7. $f(x) = e^x(\sin x - \cos x)$ in $[\frac{1}{4}\pi, \frac{5}{4}\pi]$. **[*Jabalpur* 2004]**

9. Prove that the function $f(x) = \log \dfrac{x^2 + ab}{(a+b)x}$ satisfies the conditions of Rolle's theorem in (a, b). **[*Lucknow* 04, 08, 11]**

Discuss the applicability of Rolle's theorem for these functions:

10. $f(x) = x^2$ in $[2, 3]$.

11. $f(x) = \log[(x^2+2)/3x]$ in $[1, 2]$.

12. $f(x) = \tan x$ in $[0, \pi]$.

13. Show that there is no real number k for which the equation $x^2 - 3x + k = 0$ has two distinct roots in $[0, 1]$.

14. If $\dfrac{a_0}{n+1} + \dfrac{a_1}{n} + \dfrac{a_2}{n-1} + \ldots + \dfrac{a_{n-1}}{2} + a_n = 0$,

prove that the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

has at least one root between 0 and 1. **[*Lucknow* 09, *Allahabad* 12]**

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10. Not applicable since $f(2) \neq f(3)$.

11. Applicable as $f'(x) = 0$ at $x = \sqrt{2}$ in $(1, 2)$.

12. Not applicable as f is not continuous at $x = \frac{1}{2}\pi$.

7.5. Lagrange's Mean Value Theorem. *If $f(x)$ is a function which is*

(i) *continuous in the closed interval* $[a, b]$

and (ii) *differentiable in the open interval* (a, b),

then there must be a value of x, say $c \in (a, b)$, such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

[*Kumaon* 05, *Himachal* 04, 06, *Jabalpur* 05, 08, *Avadh* 09, *Bilaspur* 06, 10, *Allahabad* 08, 11, *Kanpur* 11, *Patna* 11, *Lucknow* 05, 09, 12, *Kashi Vidyapeeth* 10, 12, *Rohilkhand* 12]

Consider the function $F(x)$ defined in such a manner that it satisfies the conditions of Rolle's theorem.

Let $$F(x)=f(x)+Cx \qquad (1)$$

where C is to be chosen such that $F(a)=F(b)$. $\qquad (2)$

From (1) and (2), we have $f(a)+Ca=f(b)+Cb$

or $$-C=\frac{f(b)-f(a)}{b-a} \qquad (3)$$

We are given that $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b). Also, Cx being a polynomial, is continuous in $[a, b]$ and differentiable in (a, b), therefore $F(x)$ is continuous in $[a, b]$ and differentiable in (a, b). Also $F(a)=F(b)$.

Thus $F(x)$ satisfies all the conditions of Rolle's theorem, so there must be at least one value of x, say $c \in (a, b)$ such that

$$F'(c)=0 \quad \text{i.e.} \quad f'(c)+C=0 \qquad \text{[from (1)]}$$

or $$f'(c)=-C \quad \text{or} \quad f'(c)=\frac{f(b)-f(a)}{b-a} \qquad \text{[using (3)]}$$

which proves the proposition. This theorem is also called ***the first mean value theorem*** of Differential Calculus.

7.51. Another form of the Mean Value Theorem. If we put $b-a=h$, then $a+\theta h$, is equal to a if $\theta=0$, and is equal to b if $\theta=1$.

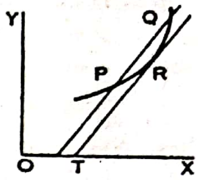
Hence $a+\theta h$, where $0<\theta<1$, means some value between a and b. So the above theorem can be written as

$$f(a+h)-f(a)=hf'(a+\theta h),$$

or $$\boldsymbol{f(a+h)=f(a)+hf'(a+\theta h)} \quad \text{where} \quad 0<\theta<1.$$

7.52. Geometrical Meaning of Lagrange's Mean Value Theorem. Let P and Q be two points on the curve $y=f(x)$. Further, let the abscissae of P and Q be a and b respectively. Then if PQ makes with the x-axis an angle ψ, we have, by geometry

$$\tan\psi=\frac{f(b)-f(a)}{b-a}.$$



Also, if the tangent to the curve at $R(x_1, y_1)$ makes an angle ψ_1 with the x- axis, we know that,

$$\tan\psi_1=f'(x_1).$$

Hence the mean value theorem merely asserts that there is some point between P and Q where the tangent is parallel to chord PQ, provided the curve has a tangent at every point between P and Q. This is geometrically obvious.

Ex. 1. Verify Lagrange's mean value theorem for the function $f(x)=\sqrt{x^2-4}$ in the interval $[2, 4]$. **[*Indore* 04, 07, *Jabalpur* 04, 09]**

Since $f(x)$ has a real and distinct value for each $x \in [2, 4]$, $f(x)$ is continuous in $[2, 4]$.

Now $$f'(x)=\tfrac{1}{2}(x^2-4)^{-1/2}(2x)=\frac{x}{\sqrt{x^2-4}}$$ which exists for each $x \in (2, 4)$ so $f(x)$ is differentiable in $(2, 4)$.

Also $$f(2)=0, \quad f(4)=\sqrt{12}=2\sqrt{3}.$$

Hence, by Lagrange's mean value theorem, there exists a $c \in (2, 4)$

such that $$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(2)}{4-2}=\frac{2\sqrt{3}-0}{2}=\sqrt{3}.$$

Now $$f'(c)=\frac{c}{\sqrt{c^2-4}}=\sqrt{3} \quad \text{i.e.} \quad c^2=3(c^2-4) \quad \text{or} \quad c=\pm\sqrt{6}$$

Discarding $c=-\sqrt{6}$ since only $c=\sqrt{6} \in (2, 4)$, the theorem is verified.

Ex. 2. Let f be defined and continuous on $[a-h, a+h]$ and derivable on $(a-h, a+h)$. Prove that there is a real number θ between 0 and 1 for which

$$f(a+h)-f(a-h)=h\{f'(a+\theta h)+f'(a-\theta h)\}.$$

Consider the function defined on $[0, 1]$ such that

$$F(x)=f(a+hx)-f(a-hx) \quad \text{where} \quad x \in [0, 1] \qquad (1)$$

Then F is continuous on $[0, 1]$ and derivable on $(0, 1)$. Hence by Lagrange's Mean Value Theorem, there exists a number θ between 0 and 1 such that

$$F(1)-F(0)=(1-0)F'(\theta)$$

i.e. $f(a+h)-f(a-h)=h\{f'(a+\theta h)+f'(a-\theta h)\}$ [using (1)]

since $F(0)=0$ and $F(1)=f(a+h)-f(a-h)$.

Ex. 3. Using Lagrange's mean value theorem, show that

$$\frac{x}{1+x}<\log(1+x)<x, \quad x>0. \qquad \text{[\textit{Delhi} 07, \textit{Bundelkhand} 11]}$$

Hence show that $0<\log(1+x)^{-1}-x^{-1}<1, \ x>0$ [*Delhi* 2008]

Let $f(x)=\log(1+x)$ in $[0, x]$

Now $f(x)$ being the logarithmic function $(x>0)$, it is continuous on $[0, x]$ and derivable on $(0, x)$, hence by Lagrange's Mean Value Theorem, there exists a number c in $(0,x)$ such that

$$\frac{f(x)-f(0)}{x-0}=f'(c) \quad \text{i.e.} \quad \frac{\log(1+x)-0}{x}=\frac{1}{1+c}$$

or
$$\log(1+x)=\frac{x}{1+c}. \tag{1}$$

Now $0<c<x$ so $1+c<1+x$ or $\frac{x}{1+x}<\frac{x}{1+c}$. (2)

Again since $0<c<x$, $\frac{x}{1+c}<x$ so using (2), we have

$$\frac{x}{1+x}<\frac{x}{1+c}<x \quad \text{i.e.} \quad \frac{x}{1+x}<\log(1+x)<x \qquad \text{[using (1)]}$$

or
$$\frac{1+x}{x}>\frac{1}{\log(1+x)}>\frac{1}{x} \quad \text{or} \quad 1+\frac{1}{x}>\frac{1}{\log(1+x)}>\frac{1}{x}$$

or
$$1>\frac{1}{\log(1+x)}-\frac{1}{x}>0 \Rightarrow 0<\log(1+x)^{-1}-x^{-1}<1 \text{ for } x>0.$$

7.6. Cauchy's Mean Value Theorem. An important extension of the mean value theorem is the following :

If (i) $f(x)$ *and* $g(x)$ *are both continuous in* $[a, b]$

(ii) $f'(x)$ and $g'(x)$ *are differentiable in* (a, b)

and (iii) $g'(x)\neq 0$ *anywhere in* (a, b),

then there is a value of x *(say c) between a and b, such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

[***Himanchal* 05, *Delhi* 05, 07, *Agra* 09, *Patna* 10, *Kumaon* 10, *Lucknow* 10, *Allahabad* 10, 13, *Raipur* 11, 13, 15**]

Consider the function F (x) defined in such a manner that it satisfies the conditions of Rolle's theorem.

Let
$$F(x)=f(x)+C\,g(x) \tag{1}$$

where C is to be chosen such that $F(a)=F(b)$. (2)

We notice that $g(b)-g(a)\neq 0$; for if $g(b)-g(a)=0$, then $g(b)=g(a)$, so that by Rolle's theorem, $g'(x)$ must vanish for some value of x between a and b, which contradicts condition (iii).

From (1) and (2), we have

$$f(a)+C\,g(a)=f(b)+C\,g(b)$$

or
$$-C=\frac{f(b)-f(a)}{g(b)-g(a)} \tag{3}$$

Since C is a constant and we are given that $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), so $F(x)$ is continuous in $[a, b]$ and differentiable in (a, b). Also $F(a)=F(b)$.

Thus $F(x)$ satisfies all the conditions of Rolle's theorem, so there must be at least one value of x, say $c\in(a, b)$ such that

$$F'(c)=0 \quad \text{i.e.} \quad f'(c)+C\,g'(c)=0 \text{ [from (1)]}$$

or
$$\frac{f'(c)}{g'(c)}=-C=\frac{f(b)-f(a)}{g(b)-g(a)}, \quad g'(x)\neq 0. \qquad \text{[using (3)]}$$

which proves the proposition. This theorem is also called ***the second mean value theorem*** of Differential Calculus.

NOTE. By putting $g(x)=x$ in the above theorem, we find that Cauchy's mean value theorem reduces to Lagrange's.

Ex. 1. If $f(x)=x^2$ and $\phi(x)=x$, verify Cauchy's mean value theorem for the given functions in the interval $[a,b]$. [*Avadh* **2010**]

Since $f(x)$ and $\phi(x)$ are polynomials, both are continuous and differentiable in $[a,b]$ for all real values of x. Also $\phi'(x)=1\neq 0$ for all $x\in(a,b)$. Hence by Cauchy's mean value theorem, for $c\in(a,b)$,

$$\frac{f'(c)}{\phi'(c)}=\frac{f(b)-f(a)}{\phi(b)-\phi(a)} \quad \text{or} \quad \frac{2c}{1}=\frac{b^2-a^2}{b-a}$$

or $c=\frac{1}{2}(b+a)$ which lies in (a,b) verifying the theorem .

Ex. 2. If in Cauchy's mean value theorem, $f(x)=1/x^2$ and $g(x)=1/x$, then c is the harmonic mean of a and b.

Now $f'(x)=-2/x^3$ and $g'(x)=-1/x^2$.

Since Cauchy's mean value theorem holds true for $f(x)$ and $g(x)$, therefore for $c \in (a,b)$, we have

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{or} \quad \frac{(-2/c^3)}{(-1/c^2)}=\frac{1/b^2-1/a^2}{1/b-1/a}$$

or $\dfrac{2}{c}=\dfrac{a^2-b^2}{ab(a-b)}$ or $c=\dfrac{2ab}{a+b}$ which is the harmonic mean of a and b.

EXAMPLES

Verify the mean value theorem for the functions:

1. $f(x)=x^3$, in $[a,b]$.
2. $f(x)=2x^2-7x+10$ in $[2, 5]$ [*Rohilkhand* 2008]
3. $f(x)=\log x$ in $[1, e]$. [*Kanpur* 2003]

Find 'c' of Lagrange's mean value theorem for :

4. $f(x)=x^3-5x^2-3x$, $a=1$, $b=3$. [*Jabalpur* 2005]
5. $f(x)=x(x-1)(x-2)$; $a=0, b=\frac{1}{2}$.
[*Jabalpur* 06, *Patna* 07, *Rohilkhand* 11, *Allahabad* 11]
6. $f(x)=(x-1)(x-2)(x-3)$ in $[0, 4]$. [*Lucknow* 2007]
7. $f(x)=\sin x\cos x$ in $[0, \pi]$.
8. $f(x)=e^x$ in $[0, 1]$.
9. Separate the intervals in which the polynomial $2x^3-15x^2+36x+1$ is increasing or decreasing. [*Delhi* 2006]
10. With the help of the mean value theorem, show that if $x>0$,
$$\log_{10}(x+1)=\frac{x\log_{10}e}{1+\theta x}, \quad \text{where} \quad 0<\theta<1.$$
[*Allahabad* 05, *Bilaspur* 09]
11. Use Lagrange's mean value theorem to prove that
$1+x<e^x<1+xe^x$, where $x>0$. [*Himachal* 2007]
12. By mean value theorem, prove that $\dfrac{x}{1+x^2}<\tan^{-1}x<x$, $x>0$.
[*Indore* 05, 06, 08, *Lucknow* 09, *Bilaspur* 10]

13. Verify Cauchy's mean value theorem when $f(x)=x^2+1$ and $g(x)=2x^3$ in $[1, 2]$. [*Jabalpur* 2008]

Find the 'c' of Cauchy's mean value theorem for the following pairs of functions :

14. $f(x)=1/x^2$, $g(x)=1/x$ in $[a,b]$. [*Bundelkhand* 2005]
15. $f(x)=\sin x$ and $g(x)=\cos x$ in $[-\frac{1}{2}\pi, 0]$.
16. $f(x)=x(x-1)(x-2)$, $g(x)=x(x-2)(x-3)$ in $[0, \frac{1}{2}]$.
[*Agra* 09, *Kumaon* 12]
17. $f(x)=e^x$ and $g(x)=e^{-x}$. [*Raipur* 12, 13, 14]
18. Show that $\dfrac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}=\cos\theta$ where $0<\alpha<\theta<\alpha<\frac{1}{2}\pi$.
[*Lucknow* 2012]
19. If in Cauchy's mean value theorem, $f(x)=\sqrt{x}$ and $\phi(x)=1/\sqrt{x}$, show that 'c' is the geometric mean between a and b.
[*Jabalpur* 2009]

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4. $\frac{7}{3}$. 5. $1-\frac{1}{6}\sqrt{21}$. 6. $2\pm 2\sqrt{3}$ 7. $\frac{1}{4}\pi, \frac{3}{4}\pi$
8. $\log_e(e-1)$. 9. Increasing in $(-\infty, 2), (3, \infty)$
and decreasing in $(2, 3)$. 14. $2ab/(a+b)$.
15. $-\frac{1}{4}\pi$. 16. $(5-\sqrt{3})/6$. 17. $\frac{1}{2}(a+b)$.

7.7. Taylor's Theorem. Finite Form. (Generalized Mean value Theorem). A more general theorem than the one of Mean Value is Taylor's theorem. The method of proof is similar to that of § 7.5. The theorem is as follows :

If $f(x)$ possesses differential coefficients of the first n orders for every value of x in the domain $[a, b]$ then

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\ldots+\frac{(b-a)^r}{r!}f^r(a)+\ldots+\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(b-a)^n}{n!}f^n\{a+\theta(b-a)\}$$

for some value of θ such that $0<\theta<1$.

Consider the function $F(x)$ given by

$$F(x) = f(b) - f(x) - (b-x)f'(x)$$
$$-\frac{(b-x)^2}{2!}f''(x) - \ldots - \frac{(b-x)^r}{r!}f^r(x) - \ldots$$
$$-\frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x) - \frac{(b-x)^n}{n!}Q$$

where Q is a constant given by the equation $F(a) = 0$, i.e. by

$$0 = f(b) - f(a) - (b-a)f'(a) - \ldots$$
$$-\{1/(n-1)!\}(b-a)^{n-1}f^{n-1}(a)$$
$$-(1/n!)(b-a)^n Q. \qquad (1)$$

Substituting b for x, we see that $F(b)$ also is zero. Now $F(x)$ consists of a finite number of terms the differential coefficients of all of which exist, and so $F'(x)$ exists for every value of x from a to b. Here we can apply Rolle's theorem, which shows that $F'(x)$ vanishes for at least one value (say x_1) of x between a and b.

Now differentiating $F(x)$ we get

$$F'(x) = -f'(x) + f'(x) - (b-x)f''(x)$$
$$+(b-x)f''(x) - \frac{(b-x)^2}{2!}f'''(x) + \ldots$$
$$-\frac{(b-x)^{n-1}}{(n-1)!}f^n(x) + \frac{(b-x)^{n-1}}{(n-1)!}Q$$

i.e. $$F'(x) = \frac{(b-x)^{n-1}}{(n-1)!}\{Q - f^n(x)\}$$

Hence the equation $F'(x_1) = 0$ becomes

$$\frac{(b-x_1)^{n-1}}{(n-1)!}\{Q - f^n(x_1)\} = 0 \qquad \text{i.e.} \qquad Q = f^{(n)}(x_1).$$

Writing $a + \theta(b-a)$ for x_1, where $0 < \theta < 1$, substituting the resulting value of Q in (1) and transposing, we get Taylor's Theorem as enunciated.

Taylor's Theorem is more usually quoted in the following form, obtained by writing x for b:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$
$$+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n}{n!}f^n\{a + \theta(x-a)\} \qquad (2)$$

But if we put h for $b - a$, and replace a by x, we get the form

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!}f^n(x)$$
$$+\frac{h^n}{n!}f^n(x + \theta h)$$

Putting $a = 0$ in (2), we get **Maclaurin's Theorem**:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}(\theta x)$$

7.71. Taylor's Series. If we denote the sum of the first n terms on the right of equation (2) above by $S_n(x)$ and the $(n+1)$th, i.e. the last term by $R_n(x)$, *Taylor's theorem can be written as*

$$f(x) = S_n(x) + R_n(x).$$

Suppose now that $f(x)$ possesses differential coefficients of all orders in a domain $(a - \lambda', a + \lambda)$ of x, so that n may be taken as large as we please without violating the conditions under which Taylor's Theorem was established. Suppose further that $R_n(x) \to 0$ as $n \to \infty$, for every value of x in $(a - \lambda', a + \lambda)$.

Now let n increase indefinitely. Then S_n becomes an infinite series and we get

$$f(x) = f(a) + (x-a)f'(a)$$
$$+\frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a) + \ldots.$$

where $$a - \lambda' < x < a + \lambda.$$

This theorem also is known as **Taylor's Theorem**.

When we want to distinguish the finite expression of the previous section from the infinite series obtained here, the former is referred to as ***Taylor's formula*** or ***Taylor's development in finite form***, and the latter as ***Taylor's series***, or ***Taylor's expansion***.

If the function to be expanded is algebraic and rational, and of degree m (say) $f^{(m+1)}(x)$ and all the succeeding differential coefficients vanish. So, the expansion is finite, and questions of convergency do not arise.

7.72. Lagrange's form of the Remainder. $R_n(x)$ is called the *Remainder after n terms*, because it is equal to the result of subtracting from $f(x)$ the sum of the first n terms of the Taylor's Series of $f(x)$. Various expressions can be found for it. The one used above, viz.,

$$R_n(x) = \frac{(x-a)^n}{n!} f^n \{a + \theta(x-a\},$$

is called *Lagrange's form of the remainder.*

7.73. Cauchy's form of the Remainder. In § 7.7, in the assumed expression for $F(x)$, if we had written the last term as $-(b-x)Q$ instead of $-(1/n!)(b-x)^n Q$, and carried out the rest of the investigation as before, the expression for $R_n(x)$ would have come out as

$$R_n(x) = \frac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!} f^n \{a + \theta(x-a)\}.$$

This gives *Cauchy's form of the remainder.*

Of course, even for the same function, the values of θ in the remainders after n terms in Cauchy's and Lagrange's forms would not be the same.

7.8. Some Expansions.

1. sin *x*. The formal series obtained by Maclaurin's Theorem in the previous chapter viz.,

$$\sin x = x - x^3/3! + x^5/5! - \ldots,$$

is convergent for every finite value of x. Moreover $\sin x$ and all its differential coefficients (which are one of the functions $\pm\sin x$, $\pm\cos x$) are numerically less than or equal to 1, whatever be the value of x.

Hence for every value of x, $R_n \le (x-a)^n/n!$, which $\to 0$ as $n \to \infty$.

Therefore $\sin x = x - x^3/3! + x^5/5! - \ldots$ for every value of x.

2. cos *x*. The expansion of this is similar to that of $\sin x$.

3. log (1 + *x*). The formal series which we get by the method of § 7.2 is convergent only if $-1 < x \le 1$. Hence we consider only these values of x.

I. $x > 0$. Lagrange's form for R_n gives

$$R_n(x) = (-1)^{n-1}(1/n)\left(\frac{x}{1+\theta x}\right)^n,$$

which $\to 0$ as $n \to \infty$, because $\{x/(1+\theta x)\}^n$ is not greater than unity, whatever n may be, when x is positive and not greater than 1.

II. $x < 0$. Cauchy's form for R_n gives

$$R_n(x) = (-1)^{n-1} x^n \cdot \frac{1}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1},$$

which $\to 0$ as $n \to \infty$, because by supposition $|x| < 1$, so $x^n \to 0$ and the other factors do not tend to infinity.

Hence the formal series is a valid expansion when $-1 < x \le 1$.

4. (1 + *x*)^m. If m is a positive integer, the expansion consists of only a finite number of terms and there is no remainder to be considered.

If m is not a positive integer, we get an infinite series which is convergent if $|x| < 1$ and divergent if $|x| > 1$. We take, therefore, $|x|$ to be less than 1. The case when $|x|$ is equal to 1 is more difficult and will not be considered here.

Cauchy's form for R_n gives

$$R_n(x) = \{1/(n-1)!\}\, x^n (1-\theta)^{n-1} . m(m-1)\ldots(m-n+1)(1+\theta x)^{m-n}$$

$$= mx(1+\theta x)^{m-1} . \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{(m-1)\ldots(m-n+1)}{(n-1)!} x^{n-1}.$$

Now $mx(1+\theta x)^{m-1}$ remains finite as $n \to \infty$, because θ always lies between 0 and 1 and so $mx(1+\theta x)^{m-1}$ lies between mx and $mx(1+x)^{m-1}$.

Also, since $|x|<1$, so $|\theta x|<\theta$ and, therefore, $1+\theta x>1-\theta$ even if θx is negative. Hence the second factor in $R_n(x)$, viz., $\{(1-\theta)/(1+\theta x)\}^{n-1}$ is not greater than unity. The last factor, viz.,

$$\frac{(m-1)(m-2)\dots(m-n+1)}{(n-1)!}x^{n-1},$$

must $\to 0$ as $n\to\infty$, because the series whose nth term is this is known to be convergent. Hence $R_n(x)\to 0$ as $n\to\infty$, and we find that the Taylor's series for $(1+x)^m$ is valid for every value of m if $|x|<1$.

Ex. 1. Prove or disprove if for $f(x)=(x-a)^{8/3}$, Taylor's expansion for $f(x)$ fails around $x=a$. [*Allahabad* 2010]

The second differential coefficient $=\frac{8}{3}\cdot\frac{5}{3}(x-a)^{2/3}$.

As the differential coefficient of $(x-a)^{2/3}$ at $x=a$ does not exist, it follows that $(x-a)^{8/3}$ is not expansible in any neighbourhood of $x=a$ by Taylor's series.

Taylor's Theorem is said to *fail* for such a case.

Ex. 2. If $f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(\theta x)$, find the value of θ as $x\to 1$, given $f(x)=(1-x)^{5/2}$. [*Lucknow* 2009]

We are given $\qquad f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(\theta x). \qquad (1)$

Since $f(x)=(1-x)^{5/2}, f'(x)=-\frac{5}{2}(1-x)^{3/2}, f'(x)=\frac{5}{2}\cdot\frac{3}{2}(1-x)^{1/2}$.

Then $\quad f(0)=1,\ f'(0)=-\frac{5}{2},\ f''(\theta x)=\frac{15}{4}(1-\theta x)^{1/2}$

Hence substituting these values in (1), we get

$$(1-x)^{5/2}=1-\tfrac{5}{2}x+\tfrac{1}{2!}x^2.\tfrac{15}{4}(1-\theta x)^{1/2}$$

Therefore as $\quad x\to 1, \quad$ we get $\quad 0=1-\frac{5}{2}+\frac{1}{2!}\cdot\frac{15}{4}(1-\theta)^{1/2}$

or $\quad (1-\theta x)^{1/2}=\frac{4}{5} \quad$ or $\quad 1-\theta=\frac{16}{25} \quad$ i.e. $\theta=\frac{9}{25}$.

Ex. 3. Show that [*Himachal* 05, *Lucknow* 07]

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+\frac{(-1)^{n-2}x^{n-1}}{n-1}+\frac{(-1)^{n-1}x^n}{n(1+\theta x)^n}.$$

By Maclaurin's theorem in the finite form, we have

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\dots$$
$$\dots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^{n}(\theta x). \qquad (1)$$

Now $\quad f(x)=\log(1+x),\quad$ then $\quad f(0)=0$.

Again $\quad f'(x)=1/(1+x), \qquad f'(0)=1$

$f''(x)=-1/(1+x)^2, \qquad f''(0)=-1=(-1)^{2-1}$

$f'''(x)=2/(1+x)^3, \qquad f'''(0)=2=(-1)^{3-1}(2!)$

$f^{iv}(x)=-3.2/(1+x)^4 \qquad f^{iv}(0)=-6=(-1)^{4-1}(3!)$

and so on till $f^{n-1}(x)=\dfrac{(-1)^{n-1}(n-2)(n-1)\dots\dots3.2.1}{(1+x)^{n-1}}=\dfrac{(-1)^{n-1}(n-2)!}{(1+x)^{n-1}}$

so that $\qquad f^{n-1}(0)=(-1)^{n-1}(n-2)!$.

Substituting these values in (1), we get

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots+\frac{(-1)^{n-2}x^{n-1}}{n-1}+\frac{(-1)^{n-1}x^n}{n(1+\theta x)^n}.$$

EXAMPLES

1. If $f(x+h)=f(x)+h\,f'(x+\theta h)$, then find the value of θ when $f(x)=x^2$. [*Lucknow* 06, 12]

2. Show that the Maclaurin expansion of e^{-1/x^2} is not valid in any interval, however small.

3. If $f(x+h)=f(x)+h\,f'(x)+\frac{h^2}{2}f''(x+\theta h)$, find the value of θ as $x\to a$, $f(x)$ being $(x-a)^{5/2}$. [*Lucknow* 2010]

4. Find by Maclaurin's Theorem the first four terms and the remainder after n terms of the expansion of $e^{ax}\cos bx$ in a series of ascending powers of x.

5. Show that $\quad e^x=1+x+\frac{x^2}{2!}+\dots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n e^{\theta x}}{n!}$.

[*Lucknow* 2010]

6. Prove that

$$\sin ax=ax-\frac{a^3x^3}{3!}+\frac{a^5x^5}{5!}-\dots+\frac{a^{n-1}x^{n-1}}{(n-1)!}\sin\left(\frac{n-1}{2}\pi\right)$$
$$+\frac{a^nx^n}{n!}\sin\left(a\theta x+\tfrac{1}{2}n\pi\right).$$ [*Bilaspur* 07, *Rohilkhand* 12]

7. Given $\quad y=f(x)=1/\sqrt{1+2x}, \quad$ prove that

$$(1+2x)\,y_{n+1}+(2n+1)\,y_n=0.$$

Expand $f(x)$ by Maclaurin's theorem with remainder after n terms. Write the remainder in both Lagrange's and Cauchy's forms.

8. Show for what values of x and at what differential coefficients Taylor's theorem will fail if

$$f(x)=\frac{(x-a)^{10}(x-b)^{13/2}(x-c)^{23/3}}{(x-d)^5}.$$

ANSWERS

1. $\frac{1}{2}$. 3. $\frac{64}{225}$.

4. $1+ax+(a^2-b^2)x^2/2!+a(a^2-3b^2)x^3/3!+\dots$.

Remainder after n terms is :

$(a^2+b^2)^{n/2}(n!)x^n e^{a\theta x}\cos\{b\theta x+n\tan^{-1}(b/a)\}$.

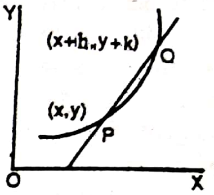
7. $(x^n/n!)f^n(\theta x)$; $\{x^n(1-\theta)^{n-1}/(n-1)!\}f^n(\theta x)$.

8. $x=b$, 7th; $x=c$, 8th; at $x=d$, the function's definition itself fails.

8

Tangents and Normals

8.1. Definition of Tangent. Let P be a given point on a curve and Q any other point on the curve near P. As Q tends to P, the straight line PQ tends, in general, to a definite straight line (whether Q be taken on one side of P or the other). This straight line is called the ***tangent*** to the curve at P.



8.11. Remark. The term "tends to" has been used here to define positions. To "Q tends to P" we can give the meaning "the value of the abscissa of Q tends to the value of the abscissa of P". Also since PQ always passes through the given point P, the definite straight line to which it tends must also pass through P, and therefore to "PQ tends to a definite straight line PT", we can assign the meaning that "the angle between PQ and the x-axis tends to the angle between PT and the x-axis."

8.12. Equation of the tangent. Let the given point P on the curve $y=f(x)$ be (x, y), and let Q, any other point on the curve, be $(x+h, y+k)$. Let the coordinates of any point R on the straight line through P and Q be (X, Y), so that (X, Y) are the current coordinates. Then the straight line PR has the equation

$$Y-y=\frac{(y+k)-y}{(x+h)-x}(X-x),$$

or

$$Y-y=\frac{f(x+h)-f(x)}{h}(X-x)$$

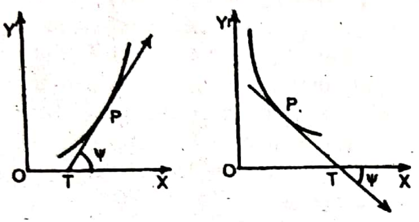
Now, as Q tends to P, this equation tends to the equation

$$Y-y=f'(x)(X-x).$$

Hence ***the tangent to the curve*** $y = f(x)$ ***at the point*** (x, y) ***is***

$$Y - y = \frac{dy}{dx}(X - x)$$

8.13. Geometrical Meaning. Let the positive direction of the tangent at P be defined as the one in which those points of the tangent lie whose abscissae are greater than that of P. Let ψ be the angle (positive, zero or negative, but not reflex), which the positive direction of the tangent at (x, y) makes with the positive direction of the x-axis (see the accompanying figures).



Comparing the equation of the tangent obtained above with the standard equation $y = mx + c$ of a straight line, we see that

$$\frac{dy}{dx} = \tan\psi,$$

i.e. the derivative dy/dx at the point (x, y) is equal to the trigonometrical tangent of the angle *which the positive direction of the tangent to the curve at (x, y) makes with the positive direction of the x-axis.*

NOTE. 1. It is clear from the figures that the ψ (and therefore dy/dx) is positive when y increases with x, and negative when y diminishes as x increases (cf. § 4.31).

2. A little consideration will show that y must lie between $+\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$.

3. The tangent of the angle (i.e. tan ψ) which the tangent at P to the curve makes with the x-axis is usually called the ***slope*** or ***gradient*** of the curve at the point P.

Ex. 1. Find the points on the curve $y = x/(1-x^2)$ where the tangent is inclined at angle of $\frac{1}{4}\pi$ to the x-axis.

Differentiating the equation of the given curve, the slope of the curve is given by

$$\frac{dy}{dx} = \frac{(1-x^2)-x(-2x)}{(1-x^2)^2} = \frac{(1+x^2)}{(1-x^2)^2}. \qquad (1)$$

But we are given that $\quad dy/dx = \tan\frac{1}{4}\pi = 1. \qquad (2)$

Equating (1) and (2), we obtain

$$1 + x^2 = (1-x^2)^2 = 1 - 2x^2 + x^4$$

or $\quad x^4 - 3x^2 = 0$ i.e. $x^2(x^2-3) = 0$ which gives $x = 0, \pm\sqrt{3}.$

Substituting these values of x in the given curve, $\quad y = 0, \mp\sqrt{3}/2$

Thus $(0, 0)$, $(\sqrt{3}, -\sqrt{3}/2)$ and $(-\sqrt{3}, \sqrt{3}/2)$ are the required points.

Ex. 2. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their point of contact lie on $x^2y^2 = x^2 - y^2$.

The given curve is $\quad y = \sin x. \qquad (1)$

Differentiating (1), $\quad dy/dx = \cos x.$

Therefore the equation of the tangent at any point (x, y) is

$$Y - y = \cos x\,(X - x) \qquad (2)$$

where (X, Y) are the current coordinates.

Since the tangent passes through the origin, we have from (2),

$$-y = -x\cos x \quad \text{or} \quad \cos x = y/x. \qquad (3)$$

The point of contact of the curve and the tangent lies on the locus of (1) and (3). Squaring and adding (1) and (3), we have

$$y^2 + y^2/x^2 = 1 \quad \text{or} \quad x^2y^2 = x^2 - y^2.$$

Ex. 3. Find the equation of the tangent at (x_1, y_1) to the ellipse $x^2/a^2 + y^2/b^2 = 1$. **[*Kumaon* 2012]**

Differentiating, $\quad \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \quad$ so that $\quad \dfrac{dy}{dx} = -\dfrac{b^2 x}{a^2 y}.$

At the point (x_1, y_1), we have $\quad dy_1/dx_1 = -b^2x_1/a^2y_1.$

and $$x_1^2/a^2+y_1^2/b^2=1. \qquad (1)$$

Therefore the equation of the tangent at (x_1, y_1) is,

$$Y-y_1=-\frac{b^2x_1}{a^2y_1}(X-x_1) \quad \text{or} \quad \frac{y_1}{b^2}(Y-y_1)+\frac{x_1}{a^2}(X-x_1)=0.$$

or $$\frac{Xx_1}{a^2}+\frac{Yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1. \qquad \text{[using (1)]}$$

where (X, Y) are the current coordinates.

Important. The equation of the tangent must in every case be simplified as much as possible. Such simplification can always be carried out in the case of algebraic curves, as in the above example.

Ex. 4. Find the equation of the tangent at the point t on the curve $x=a(t+\sin t), y=a(1-\cos t)$. **[*Kumaon* 2009 (1st part)]**

The given curve is $x=a(t+\sin t), y=a(1-\cos t)$ (1)

Differentiating (1) w.r.t t., we have

$$dx/dt=a(1+\cos t), \quad dy/dt=a\sin t.$$

So $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\sin t}{a(1+\cos t)}=\frac{2\sin\frac{1}{2}t\cos\frac{1}{2}t}{2\cos^2\frac{1}{2}t}=\tan\tfrac{1}{2}t.$$

Hence the tangent at t is

$$Y-a(1-\cos t)=\tan\tfrac{1}{2}t\,[X-a(t+\sin t)]$$

i.e. $$Y-2a\sin^2\tfrac{1}{2}t=\tan\tfrac{1}{2}t\,[X-at-2a\sin\tfrac{1}{2}t\cos\tfrac{1}{2}t]$$

i.e. $$Y-2a\sin^2\tfrac{1}{2}t=\tan\tfrac{1}{2}t\,(X-at)-2a\sin^2\tfrac{1}{2}t$$

i.e. $Y=\tan\frac{1}{2}t\,(X-at)$ which is the required equation.

8.2. Angle of intersection of curves. If two curves intersect at a point P, the angle of intersection of the curves is defined to be the angle between the tangents to the curves at P. Now the angle between the two straight lines

$$y=m_1x+c_1 \quad \text{and} \quad y=m_2x+c_2$$

is known from coordinate geometry to be

$$\tan^{-1}\{(m_1\sim m_2)/(1+m_1m_2)\}.$$

If we replace m_1 and m_2 in this formula by the values of dy/dx for the two curves at any of their points of intersection, we get at once the angle at which the curves cut there.

Ex. 1. Find the angle of intersection of the curves

$$y=4-x^2, \qquad (1)$$

and $$y=x^2. \qquad (2)$$

Subtracting (2) from (1), we see that the abscissa of the point of intersection is given by

$$4-2x^2=0 \quad \text{so that} \quad x=\sqrt{2}.$$

From (1) we find that $dy/dx=-2x=-2\sqrt{2}$ for one curve at the point of intersection.

From (2), $dy/dx=2x=2\sqrt{2}$ for the other curve at the same point.

Hence, if θ is the required angle of intersection, then

$$\tan\theta=-\frac{2\sqrt{2}+2\sqrt{2}}{1-4.2}=\frac{4\sqrt{2}}{7} \quad \text{or} \quad \theta=\tan^{-1}(4\sqrt{2}/7).$$

Ex. 2. Find the condition when the curves $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$ cut orthogonally where a, b, a_1, b_1 are all non-zero. **[*Delhi* 04, *Lucknow* 04, *Avadh* 06]**

OR

Show that the curves $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$ intersect orthogonally if $1/a-1/b=1/a_1-1/b_1$. **[*Kumaon* 2013]**

Let $P\ (x_1, y_1)$ be the point of intersection of the given curves.

Differentiating $$ax^2+by^2=1 \qquad (1)$$

We have $$2ax+2by\frac{dy}{dx}=0 \Rightarrow \left(\frac{dy}{dx}\right)_P=-\frac{ax_1}{by_1}\ (=m_1)$$

Again differentiating $a_1x^2+b_1y^2=1$ (2)

We have $$2a_1x+2b_1y\frac{dy}{dx}=0 \Rightarrow \left(\frac{dy}{dx}\right)_P=-\frac{a_1x_1}{b_1y_1}\ (=m_2)$$

For orthogonal intersection of the curves,

$$m_1m_2=-1 \Rightarrow \left(-\frac{ax_1}{by_1}\right)\left(-\frac{a_1x_1}{b_1y_1}\right)=-1$$

or $$a\,a_1x_1^2+b\,b_1y_1^2=0. \qquad (3)$$

Eliminating x_1, y_1 from (1), (2) and (3), we get

$$\begin{vmatrix} a & b & -1 \\ a_1 & b_1 & -1 \\ aa_1 & bb_1 & 0 \end{vmatrix} = 0$$

i.e. $a(bb_1) - b(aa_1) - (a_1bb_1 - aa_1b_1) = 0$

i.e. $ab(b_1 - a_1) = a_1b_1(b-a) \Rightarrow \dfrac{b_1 - a_1}{a_1b_1} = \dfrac{b-a}{ab}$

i.e. $1/a - 1/b = 1/a_1 - 1/b_1$ which is the required condition.

EXAMPLES

1. If $y = 4x^2$, then find the tangent to the curve at $(1,1)$.

[*Bundelkhand* 2006]

2. Find the equation of the tangent at the point (x, y) on each of the following curves :

(i) $y^2 = 4ax$. (ii) $x^m / a^m + y^m / b^m = 1$.

(iii) $xy = a^2$. (iv) $y = a \cosh(x/a)$.

(v) $y = a \log \sin x$.

3. Find the points at which the tangent to each of the following curves is (*a*) parallel to and (*b*) perpendicular to the axis of x:

(i) $ax^2 + 2hxy + by^2 = 1$. (ii) $y = x^{2/3}(x+a)^{1/3}$

(iii) $y = x^2 + \sqrt{1 - x^2}$

[HINT: The tangent is perpendicular to the x-axis if $dx/dy = 0$.]

4. Find the equation of the tangent at the point t on each of the following curves:

(i) $x = t^2 - a, y = t^3 - b$. (ii) $x = a \sin^3 t, y = b \cos^3 t$.

5. Prove that $x/a + y/b = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y. [*Sagar* 2002]

6. Prove that the curve $(x/a)^n + (y/b)^n = 2$ touches the straight line $x/a + y/b = 2$ at the point (a, b) whatever be the value of n.

7. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again at (X, Y), then show that $\dfrac{X}{x_1} + \dfrac{Y}{y_1} = 1$. [*Magadh* 2005]

8. Show that the tangents to the Folium of Descartes $x^3 + y^3 = 3axy$ at the points where it meets the parabola $y^2 = ax$ are parallel to the axis of y.

9. Find the condition that the straight line $p = x \cos\alpha + y \sin\alpha$ touches the curve $\left(\dfrac{x}{a}\right)^{n/(n-1)} + \left(\dfrac{y}{b}\right)^{n/(n-1)} = 1$. [*Patna* 2004]

10. Show that the line $x\cos\theta + y\sin\theta - p = 0$ will touch the curve $x^m y^n - a^{m+n} = 0$ if $p^{m+n}.m^m.n^n = (m+n)^{m+n} a^{m+n} \cos^m\theta \sin^n\theta$.

[*Delhi* 03, *Patna* 06]

11. Find the angle of intersection between the curves

(i) $2y^2 = x^3$ and $y^2 = 32x$. (ii) $x^2 = 4ay$ and $2y^2 = ax$.

(iii) $xy = a^2$ and $x^2 + y^2 = 2a^2$. (iv) $y = a^x$ and $y = b^x$.

(v) $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$.

12. Find the angle of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at their point of intersection other than the origin.

ANSWERS

1. $y - 8x + 7 = 0$.

2. (i) $yY = 2a(X + x)$. (ii) $y^{m-1}Y/b^m + x^{m-1}X/a^m = 1$

(iii) $Y/y + X/x = 2$. (iv) $Y - y = \{\sinh(x/a)\}(X - x)$

(v) $Y - y = a(\cot x)(X - x)$.

3. (i) (a) where $ax + hy = 0$ intersects the curve.

(b) where $by + hx = 0$ intersects the curve.

(ii) (a) at $(-2a/3, 4^{1/3}a/3)$; (b) at $(0, 0), (-a, 0)$;

(iii) (a) at $(0, \pm 1), (\pm\sqrt{3}/2, \frac{5}{4})$; (b) at $(\pm 1, 1)$.

4. (i) $2Y = 3tX - t^3 + 3at - 2b$.

(ii) $X/a \sin t + Y/b \cos t = 1$.

9. $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$.

11. (i) $\frac{1}{2}\pi$ and $\tan^{-1}\frac{1}{2}$. (ii) $\frac{1}{2}\pi$ and $\tan^{-1}\frac{3}{5}$

(iii) 0 i.e. the curves touch each other.

11. (iv) $\dfrac{\log a/b}{1+\log a \log b}$ (v) $\frac{1}{4}\pi$. **12.** $\tan^{-1}\dfrac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$

8.3. Normal. The normal to a curve at any point is the straight line which passes through that point and is at right angles to the tangent to the curve at that point.

Any line through the point (x, y) is

$$Y - y = m(X - x).$$

This will be perpendicular to the tangent to the curve at (x, y) if

$$m.\frac{dy}{dx} = -1.$$

Substituting the value of m obtained from this in the first equation, we see that ***the normal at (x, y) to the curve*** $y = f(x)$ ***is***

$$\frac{dy}{dx}(Y - y) + X - x = 0.$$

Ex. 1. Find the equation of the normal at (x, y) to the ellipse

$$x^2/a^2 + y^2/b^2 = 1.$$

Differentiating, we get $\dfrac{dy}{dx} = -\dfrac{b^2 x}{a^2 y}$ [see § 8.13, Ex.3]

Hence the normal at (x, y) is $-(b^2 x/a^{2y})(Y - y) + X - x = 0$

or

$$\frac{X - x}{x/a^2} = \frac{Y - y}{y/b^2}.$$

Ex. 2. If $lx + my = 1$ is a normal to the parabola $y^2 = 4ax$, then prove that $al^3 + 2alm^2 = m^2$. **[*Lucknow* 08, 12]**

Differentiating $y^2 = 4ax$, we have

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}.$$

Therefore the normal at (x, y) to the parabola is

$$(2a/y)(Y - y) + X - x = 0$$

or

$$2aY + yX = (2a + x)y.$$

If this is the same as $lX + mY = 1$, then

$$\frac{y}{l} = \frac{2a}{m} = \frac{(2a + x)y}{1}.$$

This gives $y = 2al/m$ and $2a + x = 1/l$.

Putting these values in $y^2 = 4ax$, we get

$$(2al/m)^2 = 4a(1/l - 2a) \quad \text{or} \quad 4a^2l^2.l = 4am^2(1 - 2al)$$

i.e. $al^3 + 2alm^2 = m^2$ which proves the proposition.

EXAMPLES

1. Find the equation of the normal at (x, y) on each of the following curves :

(i) $y^2 = 4ax$. (ii) $x^m/a^m + y^m/b^m = 1$.

(iii) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ (iv) $y = a \log \sin x$.

2. Find the equations of the tangent and the normal to the curve $y(x - 2)(x - 3) - x + 7 = 0$ at the point where it cuts the axis of x.

3. Find the equation of the normal at $\left(cp, \dfrac{c}{p}\right)$ to the curve $xy = c^2$. **[*Delhi* 2005]**

4. Find the equation of the normal at any point on the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ and show that, at the point where $\theta = \pi/4$, the normal passes through the origin.

5. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is

$$y\cos\phi - x\sin\phi = a\cos 2\phi \qquad \textbf{[\textit{Delhi} 2002]}$$

6. Show that the normal at any point of the curve

$$x = a\cos\theta + a\theta\sin\theta, \quad y = a\sin\theta - a\theta\cos\theta$$

is at a constant distance from the origin.

7. Show that the tangent and normal at any point of the curve

$$x = ae^{\theta}(\sin\theta - \cos\theta), \quad y = ae^{\theta}(\sin\theta + \cos\theta)$$

are equidistant from the origin

8. Show that the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ touches the curve $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

9. If the tangent at any point $(4m^2, 8m^3)$ of $x^3 = y^2$ is a normal to the curve at the point where it meets the curve again, then find the value of m.

ANSWERS

1. (i) $2aY + yX = xy + 2aY$.

   (ii) $y^{m-1}X/b^m - x^{m-1}Y/a^m = xy^{m-1}/b^m - yx^{m-1}/a^m$.

   (iii) $(2x^2 + 2y^2 + a^2)yX - (2x^2 + 2y^2 - a^2)xY = 2a^2xy$.

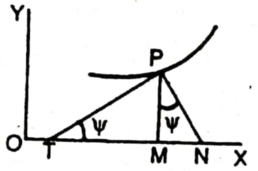
   (iv) $X\sin x + aY\cos x = x\sin x + ay\cos x$.

2. $20y - x + 7 = 0,\ y + 20x - 140 = 0$.

3. $xp^3 - yp + c(1 - p^4) = 0$.

4. $y\cos^3\theta - x\sin^3\theta = \frac{3}{4}\sin 4\theta$. 9. $m = \sqrt{2/3}$.

8.31. Cartesian Subtangent and subnormal. Let the tangent and the normal at any point P on a curve meet the x-axis in T and N respectively, and let PM be the ordinate. Then TM is called the ***subtangent*** and MN the ***subnormal***.



If the angle which the tangent makes with the x-axis be ψ, we have $\tan\psi = dy/dx$ and $MP = y$. Therefore

$$\textbf{Subtangent} = TM = y\cot\psi = y\bigg/\left(\frac{dy}{dx}\right)$$

$$\textbf{Subnormal} = MN = y\tan\psi = y\frac{dy}{dx}.$$

Ex. 1. Prove that the subnormal at any point of a parabola is of constant length and the subtangent varies as the abscissa of the point of contact.

Let the parabola be $y^2 = 4ax$.

Then $2y\,dy/dx = 4a$ or $dy/dx = 2a/y$.

Therefore, the subnormal $= y\dfrac{dy}{dx} = y\left(\dfrac{2a}{y}\right) = 2a$,

which is of constant length; and

$$\text{the subtangent} = y\bigg/\frac{dy}{dx} = y\left(\frac{y}{2a}\right) = \frac{4ax}{2a} = 2x,$$

which proves the proposition.

Ex. 2. Show that for the curve $by^2 = (x+a)^3$, the square of the subtangent varies as the subnormal. [*Lucknow* **2012**]

The given equation is $by^2 = (x+a)^3$. (1)

Then $2by\dfrac{dy}{dx} = 3(x+a)^2$ or $\dfrac{dy}{dx} = \dfrac{3(x+a)^2}{2by}$.

$$\text{Hence the subnormal} = y\frac{dy}{dx} = y\cdot\frac{3(x+a)^2}{2by} = \frac{3(x+a)^2}{2b}.$$

$$\text{Also, the subtangent} = y\bigg/\frac{dy}{dx} = y\cdot\frac{2by}{3(x+a)^2} = \frac{2by^2}{3(x+a)^2}$$

$$= \frac{2(x+a)^3}{3(x+a)^2} = \frac{2}{3}(x+a). \qquad \text{[using (1)]}$$

Therefore the ratio of the square of the subtangent to the subnormal

$$= \frac{(\text{subtangent})^2}{\text{subnormal}} = \frac{4}{9}(x+a)^2\bigg/\frac{3(x+a)^2}{2b} = \frac{8b}{27} = \text{constant}$$

which proves the proposition.

8.32. Intercepts. The equation of the tangent is

$$Y - y = (dy/dx)(X - x).$$

This meets the x-axis OX where $Y = 0$, i.e.

$$0 - y = \frac{dy}{dx}(X - x), \quad \text{or} \quad X = x - y\bigg/\frac{dy}{dx}.$$

Hence the intercept which the tangent cuts off from the axis of x is $\quad x - y\bigg/\dfrac{dy}{dx}$.

This intercept in the figure of the previous article is OT. Similarly, by putting $X = 0$ in the equation of the tangent, we see that the intercept which the tangent cuts off from the axis of y is $\quad y - x\dfrac{dy}{dx}$.

The length of the tangent intercepted between the point of contact and the axis of x is often called the ***length of the tangent.*** In the figure, this is PT. So from the right-angled triangle PTM, we have

$$PT = PM \operatorname{cosec}\psi = y \operatorname{cosec}\psi = y\sqrt{1+\cot^2\psi}$$

Therefore, **Tangent** $= PT = y\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$

Similarly, **Normal** $= PN = y\sec\psi = y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

where PN is the ***length of the normal.***

8.33. Sign. In the figure of § 8.31, y and dy/dx are both positive. Hence $y/(dy/dx)$ will be positive in such a case. Also, T is to the left of M. Hence, if we adhere to the usual convention that distances on OX are to be taken as positive when measured in the direction OX, it is evident that ***the subtangent must be measured positively from T in the direction OX. If its value comes out to be negative, we must infer that M lies to the left of T.*** A similar interpretation is possible in the case of the subnormal and the intercepts.

Ex. 1. Prove that the sum of intercepts on the coordinate axes of any tangent to $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is constant. ***[Delhi* 2006]**

OR If the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at any point on it, cuts the x-axis at P and the y-axis at Q, then prove that $OP+OQ=a$, where O is the origin. ***[Lucknow* 2009]**

The given curve is $\quad \sqrt{x}+\sqrt{y}=\sqrt{a}. \quad (1)$

Differentiating, $\quad \dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}=0 \quad$ or $\quad \dfrac{dy}{dx}=-\dfrac{\sqrt{y}}{\sqrt{x}}$.

The equation of the tangent to (1) at any point (x, y) is

$$Y-y=(-\sqrt{y}/\sqrt{x})(X-x)$$

where (X, Y) are the current coordinates.

The tangent (2) meets the x-axis where $Y=0$ so that

$$-y=(-\sqrt{y}/\sqrt{x})(X-x)$$

or $\quad X=x+\sqrt{xy}=\sqrt{x}\,(\sqrt{x}+\sqrt{y})=\sqrt{x}\sqrt{a}.\quad$ [using (1)] $\quad (3)$

which is the intercept on the x-axis.

Also the tangent (2) meets the y-axis where $X=0$. Then

$$Y=y+(\sqrt{y}/\sqrt{x}).x=y+\sqrt{xy}=\sqrt{y}\,(\sqrt{x}+\sqrt{y})=\sqrt{y}\sqrt{a} \qquad (4)$$

again using (1). This is the intercept on the y-axis.

Adding (3) and (4), the sum of intercepts on the coordinate axes is $(\sqrt{x}\sqrt{a}+\sqrt{y}\sqrt{a})=\sqrt{a}(\sqrt{x}+\sqrt{y})=\sqrt{a}\sqrt{a}=a$, a constant.

Ex. 2. Show that length of the portion of the tangent to the curve $x=a\cos^3\theta$, $y=a\sin^3\theta$ intercepted between the coordinate axes is constant. ***[Delhi* 2005]**

Differentiating the equation of the given curve, we have

$$dx/d\theta=-3a\cos^2\theta\sin\theta,\quad dy/d\theta=3a\sin^2\theta\cos\theta.$$

Therefore $\quad \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta$.

So the equation of the tangent to the given curve is

$$Y-a\sin^3\theta=-\tan\theta\,(X-a\cos^3\theta)$$

or $\quad X\sin\theta+Y\cos\theta-a\sin\theta\cos\theta=0, \quad (1)$

where (X, Y) are the current coordinates.

The tangent meets x-axis where $Y=0$, then from (1), $X=a\cos\theta$, i.e. the tangent meets x-axis at the point $(a\cos\theta, 0)$.

Similarly, from (1) when $X=0$, the tangent meets y-axis at the point $(0, a\sin\theta)$.

Hence the length of the portion of the tangent intercepted between the coordinate axes i.e. between the points $(a\cos\theta, 0)$ and $(0, a\sin\theta)$

$$=\sqrt{(a\cos\theta-0)^2+(0-a\sin\theta)^2}=\sqrt{a^2\cos^2\theta+a^2\sin^2\theta}=a,$$

which is a constant. This proves the proposition.

EXAMPLES

1. Show that in the exponential curve $y=be^{x/a}$, the subtangent is of constant length and the subnormal varies as the square of the ordinate.

2. Find the length of the normal and subnormal of the curve $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$.

3. In the curve $x^{m+n} = a^{m+n} y^{2n}$, prove that the mth power of the subtangent varies as the nth power of the subnormal.

4. Prove that the subnormal at any point of the curve $y^2x^2 = a^2(x^2 - a^2)$ varies inversely as the cube of its abscissa.

5. Find the subtangent, subnormal, normal, tangent, and the intercept on the axes at the point t on the cycloid

$$x = a(t + \sin t), \quad y = a(1 - \cos t).$$

6. What should be the value of n in the equation to the curve $y = a^{1-n} x^n$ in order that the subnormal may be of constant length ?

7. Show that length of the tangent to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ intercepted between the coordinate axes is constant. [*Delhi* 03, 07]

8. In the tractrix $x = a(\cos t + \log \tan \frac{1}{2}t)$, $y = a \sin t$,

prove that portion of the tangent intercepted between the curve and the axis of x is of constant length.

9. Find the abscissa of the point on the curve $ay^2 = x^3$ at which the normal cuts off equal intercepts from the coordinate axes.

10. Prove that portion of the tangent to the curve $x^m y^n = a^{m+n}$ intercepted between the axes is divided in the ratio $m:n$ at the point of contact.

11. Prove that for the catenary $y = c\cosh(x/c)$, the perpendicular dropped from the foot of the ordinate upon the tangent is of constant length.

12. Prove that in the ellipse $x^2/a^2 + y^2/b^2 = 1$, the length of the normal varies inversely as the perpendicular from the origin on the tangent.

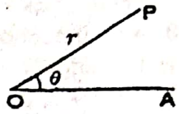
ANSWERS

2. $a\cosh^2(x/a)$, $\frac{1}{2}a\sinh(2x/a)$.

5. $a\sin t$; $2a\sin^3\frac{1}{2}t\sec\frac{1}{2}t$; $2a\sin\frac{1}{2}t\tan\frac{1}{2}t$; $2a\sin\frac{1}{2}t$; at; $-at\tan\frac{1}{2}t$. 6. 1/2.

8.4. Polar Coordinates. The position of a point P on a plane can also be indicated by stating (i) its distance r from a fixed point O, and (ii) the inclination θ of OP to a fixed straight line OA. r and θ are called the ***polar coordinates*** of

P; r is called ***the radius vector***, θ ***the vectorial angle***, O ***the pole***, and OA ***the initial line***. r is considered to be positive when measured away from O along the line bounding the vectorial angle, and θ is considered to be positive when measured in the anti-clockwise direction.



It is usual to regard θ as the independent variable.

When converting polar coordinates to Cartesians, or vice versa, it is customary to take the pole as the origin and the initial line as the x-axis. Then the formulae of conversion are

$$x = r\cos\theta, \quad y = r\sin\theta$$

and conversely, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$.

If r and θ are given, there is only one point which will have the coordinates (r, θ). But if P be given, not only may the coordinates be started as (r, θ), but also as $(r, \theta \pm 2\pi)$, $(r, \theta \pm 4\pi)$, ... ,or as $(-r, \theta \pm \pi)$, $(-r, \theta \pm 3\pi)$,.... The student should remember this, otherwise he would commit mistakes.

8.5. Angle between Radius Vector and Tangent. Let P be a given point on the curve $r = f(\theta)$ and let Q be any neighbouring point on the curve.

Let P be the point (r, θ) and Q the point $(r + \delta r, \theta + \delta\theta)$*.

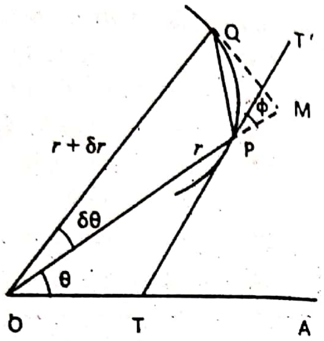
Let TPT' be the tangent at P, and let ϕ be the angle between the tangent and the radius vector OP. We have to find ϕ.

Draw QM perpendicular to OP (produced if necessary).

---

* In the equation $r = f(\theta)$, r and θ are current coordinates; but when we say that the coordinates of the given point are r and θ, we attach a different meaning to these symbols. It would be best to use (r_1, θ_1) for the coordinates of P. But this is inconvenient. However, after this explanation the student should have no difficulty in understanding whether at any given place the reference is to the coordinates of P or to the current coordinates.

Then as $Q \to P$, $\delta\theta \to 0$, the chord $PQ \to$ the tangent PT' and the $\angle QPM \to \angle\phi$.



Hence $\tan\phi$

$$= \tan \lim_{\delta\theta\to 0} \angle QPM$$

$$= \lim_{\delta\theta\to 0} \tan \angle QPM$$

$$= \lim_{\delta\theta\to 0} QM / PM$$

$$= \lim_{\delta\theta\to 0} \frac{QM}{OM - OP}$$

$$= \lim_{\delta\theta\to 0} \frac{(r+\delta r)\sin\delta\theta}{(r+\delta r)\cos\delta\theta - r} = \lim_{\delta\theta\to 0} \frac{(r+\delta r)\,\delta\theta}{(r+\delta r)\cdot 1 - r}$$

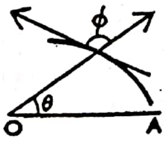
since as $\delta\theta \to 0$, $\sin\delta\theta \to \delta\theta$ and $\cos\delta\theta \to 1$.

$$= \lim_{\delta\theta\to 0} \frac{r\,\delta\theta + \delta r\,\delta\theta}{\delta r} = \lim_{\delta\theta\to 0} r\frac{\delta\theta}{\delta r} = r\frac{d\theta}{dr}$$

on neglecting $\delta r\,\delta\theta$ which is a very small quantity of the second order.

Thus $\boxed{\tan\phi = r\dfrac{d\theta}{dr}}.$

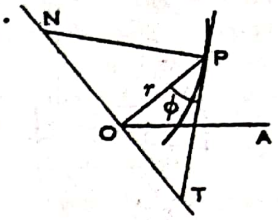
If we suppose ϕ to be numerically less than π and define it to be the angle between the positive direction of the radius vector and that direction of the tangent in which θ increases, it is easy to see that the above formula holds whether $r\,d\theta/dr$ be positive or negative. If $r\,d\theta/dr$ is negative, it means that ϕ is greater than $\frac{1}{2}\pi$ (See the marginal figure).



8.6. Polar Subtangent and Subnormal. Let P be any point on a curve $r = f(\theta)$, and let the tangent and normal at P meet the straight line through the pole at right angles to the radius vector OP in T and N respectively.

Then OT is called the ***polar subtangent*** and ON the ***polar subnormal***.

Since $\angle OPT = \phi$, $OT = r\tan\phi$.



Hence the **polar subtangent**

$$= r^2\frac{d\theta}{dr}.$$

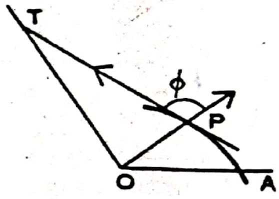
Also, $ON = r\tan\angle OPN$

$$= r\tan(\tfrac{1}{2}\pi - \phi) = r\cot\phi.$$

Hence the **polar subnormal**

$$= \frac{dr}{d\theta}.$$

The formula for the polar subtangent shows that it is measured positively to the right, the observer being supposed to be stationed at O and looking in the direction of P. A negative value of the subtangent shows that T is to the left, as in the marginal figure. A similar meaning can be assigned to a negative subnormal.



We shall now find the lengths of the polar tangent PT and polar normal PN at P.

Again, from $\Delta\, OPT$, $PT / r = \sec\phi$

or $$PT = r\sec\phi = r\sqrt{1+\tan^2\phi} = r\sqrt{1 + r^2(d\theta/dr)^2}.$$

So **the polar tangent** $= r\sqrt{1 + r^2(d\theta/dr)^2}$.

Also from ΔOPN,

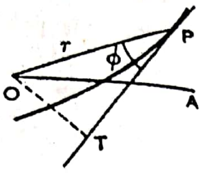
$$PN / r = \sec\angle OPN = \sec(\tfrac{1}{2}\pi - \phi) = \operatorname{cosec}\phi$$

or $$PN = r\operatorname{cosec}\phi = r\sqrt{1+\cot^2\phi} = r\sqrt{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{r^2 + (dr/d\theta)^2}.$$

So **the polar normal** $= \sqrt{r^2 + (dr/d\theta)^2}$.

8.61. Perpendicular from Pole on Tangent. If p be the length of the perpendicular OT from the pole O to the tangent at any point P on the curve, then, from the figure.



$$\boxed{p = r \sin\phi} \qquad (1)$$

If we want the result in terms of r and θ, we have only to substitute for $\sin\phi$ from the equation $\cot\phi = (1/r)\, dr/d\theta$. Thus, from (1),

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\},$$

or
$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.} \qquad \text{[\textit{Patna} 07, \textit{Magadh} 10]}$$

Sometimes u is used to denote $1/r$, then $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$ and this formula can be stated as

$$\mathbf{1/p^2 = u^2 + (du/d\theta)^2.} \qquad \text{[\textit{Patna} 05, \textit{Magadh} 07]}$$

Ex. 1. For the curve $r = a(1-\cos\theta)$, show that

(i) $\phi = \frac{1}{2}\theta$ [*Rohilkhand* 03, *Lucknow* 11]

(ii) $2ap^2 = r^3$ (iii) Polar subtangent $= 2a\sin^2\frac{1}{2}\theta\tan\frac{1}{2}\theta$.

[*Rohilkhand* 03]

The equation of the given curve is $r = a(1-\cos\theta)$. (1)

(i) Differentiating (1), $dr/d\theta = a\sin\theta$. (2)

Therefore $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a(1-\cos\theta)}{a\sin\theta} = \dfrac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$

or $\tan\phi = \tan\frac{1}{2}\theta$ which gives $\phi = \frac{1}{2}\theta$.

(ii) We have $p = r\sin\phi = r\sin\frac{1}{2}\theta$. (3)

Substituting (3) in (1), we have

$$r = a(1-\cos\theta) = 2a\sin^2\tfrac{1}{2}\theta = 2a(p/r)^2$$

or
$$r^3 = 2ap^2.$$

(iii) The polar subtangent

$$= r^2\frac{d\theta}{dr} = \frac{r^2}{dr/d\theta} = \frac{a^2(1-\cos\theta)^2}{a\sin\theta} \qquad \text{[using (1) \& (2)]}$$

$$= \frac{a(2\sin^2\frac{1}{2}\theta)}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = 2a\sin^2\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta.$$

Ex. 2. Show that for the curve $r^2 = a^2\cos 2\theta$, the length of the polar normal varies inversely as the radius vector. [*Lucknow* 2008]

The equation of the given curve is $r^2 = a^2\cos 2\theta$ (1)

Differentiating (1) w.r.t. θ $2r\, dr/d\theta = -2a^2\sin 2\theta$

i.e.
$$dr/d\theta = (-a^2\sin 2\theta)/r \qquad (2)$$

The polar normal $= \sqrt{r^2 + (dr/d\theta)^2}$

$$= \sqrt{r^2 + \frac{a^4\sin^2 2\theta}{r^2}} = \frac{1}{r}\sqrt{r^4 + a^4\sin^2 2\theta} \qquad \text{[using (2)]}$$

$$= \frac{1}{r}\sqrt{a^4\cos^2 2\theta + a^4\sin^2 2\theta} = \frac{a^2}{r} \qquad \text{[from (1)]}$$

which shows that the polar normal varies inversely as r.

8.7. Pedal equation. The relation between p and r for a given curve is called its pedal equation. For certain curves the equation is very simple.

(i) ***To find the pedal equation from the Cartesian equation.*** The tangent at (x, y) is

$$Y - y - (X - x)\frac{dy}{dx} = 0.$$

Hence, if the length of the perpendicular on it from the origin $(X = 0, Y = 0)$ is p, we have

$$p = \frac{x(dy/dx) - y}{\sqrt{1 + (dy/dx)^2}} \qquad (1)$$

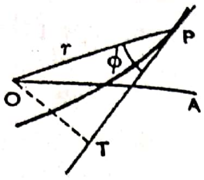
Also, if r be the distance of between the origin and any point (x, y) on the curve, then $r^2 = x^2 + y^2$ (2)

If the equation to the curve is known, say

$$f(x, y) = 0. \qquad (3)$$

Then we can get the pedal equation by eliminating x and y between the equations (1), (2) and (3).

8.61. Perpendicular from Pole on Tangent. If p be the length of the perpendicular OT from the pole O to the tangent at any point P on the curve, then, from the figure.



$$\boxed{p = r\sin\phi} \qquad (1)$$

If we want the result in terms of r and θ, we have only to substitute for $\sin\phi$ from the equation $\cot\phi = (1/r)\,dr/d\theta$. Thus, from (1),

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\},$$

or
$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.} \qquad \text{[\textit{Patna} 07, \textit{Magadh} 10]}$$

Sometimes u is used to denote $1/r$, then $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$ and this formula can be stated as

$$\mathbf{1/p^2 = u^2 + (du/d\theta)^2.} \qquad \text{[\textit{Patna} 05, \textit{Magadh} 07]}$$

Ex. 1. For the curve $r = a(1-\cos\theta)$, show that

(i) $\phi = \frac{1}{2}\theta$ **[*Rohilkhand* 03, *Lucknow* 11]**

(ii) $2ap^2 = r^3$ (iii) Polar subtangent $= 2a\sin^2\frac{1}{2}\theta\tan\frac{1}{2}\theta$.

[*Rohilkhand* 03]

The equation of the given curve is $r = a(1-\cos\theta)$. (1)

(i) Differentiating (1), $dr/d\theta = a\sin\theta$. (2)

Therefore $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a(1-\cos\theta)}{a\sin\theta} = \dfrac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$

or $\tan\phi = \tan\frac{1}{2}\theta$ which gives $\phi = \frac{1}{2}\theta$.

(ii) We have $p = r\sin\phi = r\sin\frac{1}{2}\theta$. (3)

Substituting (3) in (1), we have

$$r = a(1-\cos\theta) = 2a\sin^2\tfrac{1}{2}\theta = 2a(p/r)^2$$

or
$$r^3 = 2ap^2.$$

(iii) The polar subtangent

$$= r^2\frac{d\theta}{dr} = \frac{r^2}{dr/d\theta} = \frac{a^2(1-\cos\theta)^2}{a\sin\theta} \qquad \text{[using (1) \& (2)]}$$

$$= \frac{a(2\sin^2\frac{1}{2}\theta)}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = 2a\sin^2\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta.$$

Ex. 2. Show that for the curve $r^2 = a^2\cos 2\theta$, the length of the polar normal varies inversely as the radius vector. **[*Lucknow* 2008]**

The equation of the given curve is $r^2 = a^2\cos 2\theta$ (1)

Differentiating (1) w.r.t. θ $2r\,dr/d\theta = -2a^2\sin 2\theta$

i.e. $dr/d\theta = (-a^2\sin 2\theta)/r$ (2)

The polar normal $= \sqrt{r^2 + (dr/d\theta)^2}$

$$= \sqrt{r^2 + \frac{a^4\sin^2 2\theta}{r^2}} = \frac{1}{r}\sqrt{r^4 + a^4\sin^2 2\theta} \qquad \text{[using (2)]}$$

$$= \frac{1}{r}\sqrt{a^4\cos^2 2\theta + a^4\sin^2 2\theta} = \frac{a^2}{r} \qquad \text{[from (1)]}$$

which shows that the polar normal varies inversely as r.

8.7. Pedal equation. The relation between p and r for a given curve is called its pedal equation. For certain curves the equation is very simple.

(i) *To find the pedal equation from the Cartesian equation.* The tangent at (x, y) is

$$Y - y - (X - x)\frac{dy}{dx} = 0.$$

Hence, if the length of the perpendicular on it from the origin $(X = 0, Y = 0)$ is p, we have

$$p = \frac{x(dy/dx) - y}{\sqrt{1 + (dy/dx)^2}} \qquad (1)$$

Also, if r be the distance of between the origin and any point (x, y) on the curve, then $r^2 = x^2 + y^2$ (2)

If the equation to the curve is known, say

$$f(x, y) = 0. \qquad (3)$$

Then we can get the pedal equation by eliminating x and y between the equations (1), (2) and (3).

(ii) ***To find the pedal equation from the polar equation.*** Let the equation to the curve be

$$f(r, \theta) = 0. \tag{1}$$

Also, by the previous article,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \tag{2}$$

We get the pedal equation by eliminating θ between (1) and (2).

Another method of finding the pedal equation of the curve is to eliminate θ and ϕ from (1) and (3) where

$$p = r\sin\phi \qquad \tan\phi = r\frac{d\theta}{dr} \tag{3}$$

Ex. 1. Find the pedal equation of the parabola $y^2 = 4a(x + a)$.

[*Guwahati* 05, *Meerut* 10, *Lucknow* 05, 11]

Differentiating the equation of the parabola, we have

$$2y\frac{dy}{dx} = 4a \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2a}{y}.$$

Hence the tangent at (x, y) is $Y - y = (2a/y)(X - x)$.

Therefore $$p = \frac{x(2a/y) - y}{\sqrt{1 + 4a^2/y^2}} = \frac{2ax - y^2}{\sqrt{y^2 + 4a^2}}$$

$$= \frac{2ax - 4a(x + a)}{\sqrt{4ax + 4a^2 + 4a^2}} = \frac{-2a(x + 2a)}{\sqrt{4a(x + 2a)}}$$

$$= -\sqrt{a(x + 2a)} = -\sqrt{ar}$$

since $$r^2 = x^2 + y^2 = x^2 + 4a(x + a) = (x + 2a)^2.$$

Therefore $p^2 = ar$ which is the required pedal equation.

Ex. 2. Find the pedal equation of the cardioid $r = a(1 + \cos\theta)$.

[*Delhi* 04, *Calcutta* 05]

Here $$r = a(1 + \cos\theta) \tag{1}$$

Therefore $$dr/d\theta = -a\sin\theta$$

We have $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}.a^2\sin^2\theta = \frac{1}{r^2} + \frac{a^2}{r^4}(1 - \cos^2\theta)$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4}\left[1 - \left(\frac{r}{a} - 1\right)^2\right] \qquad \text{[using (1)]}$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4}\left(\frac{2r}{a} - \frac{r^2}{a^2}\right) = \frac{2a}{r^3}$$

i.e. $2ap^2 = r^3$ is the required pedal equation.

Ex. 3. Find the pedal equations of the curve $r^n = a^n \cos n\theta$.

[*Lucknow* 08, *Patna* 08, 10, *Magadh* 09, 11]

Taking the logarithm of the given equation, we have

$$n\log r = n\log a + \log\cos n\theta.$$

Differentiating this with respect to θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n\sin n\theta) = -n\tan n\theta \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta.$$

Now $$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta = \cot\left(\tfrac{1}{2}\pi + n\theta\right),$$

which gives $\phi = \frac{1}{2}\pi + n\theta$.

But $p = r\sin\phi = r\sin\left(\frac{1}{2}\pi + n\theta\right) = r\cos n\theta = r(r^n/a^n) = r^{n+1}/a^n$.

Hence the required pedal equation is $pa^n = r^{n+1}$.

8.8. Angle of intersection of curves. If two curves, whose polar equations are known, intersect at P, and the values of ϕ at that point for the two curves are ϕ_1 and ϕ_2 respectively, then the angle of intersection of the two curves, say α, is evidently $\phi_1 \sim \phi_2$.

If $\tan\phi_1 = m_1$ and $\tan\phi_2 = m_2$, then

$$\tan\alpha = \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 - \tan\phi_1\tan\phi_2}.$$

Hence the angle of intersection of the two curves is

$$\alpha = \tan^{-1}\left(\frac{\tan\phi_1 - \tan\phi_2}{1-\tan\phi_1 \tan\phi_2}\right).$$

In particular, the ***curves cut orthogonally*** (***i.e. at right angles***) ***if*** $m_1 m_2 = -1$.

Ex. Find the angle of intersection of the curves $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$. [*Kanpur* 07, 11, *Lucknow* 12]

OR. Show that the curves $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$ intersect orthogonally. [*Delhi* 07, *Meerut* 11]

Let P be the point of intersection of the two curves. Let ϕ_1, ϕ_2 be the angles which the the two tangents to the curves at P make with the radius vector OP.

For the curve $r = a(1+\cos\theta)$, $dr/d\theta = -a\sin\theta$.

Then $$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{-2\cos^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = -\cot\tfrac{1}{2}\theta$$

or $$\tan\phi_1 = \tan\left(\tfrac{1}{2}\pi+\tfrac{1}{2}\theta\right) \quad \text{so that} \quad \phi_1 = \left(\tfrac{1}{2}\pi+\tfrac{1}{2}\theta\right) \qquad (1)$$

Again, for the curve $r = b(1-\cos\theta)$, $dr/d\theta = b\sin\theta$.

So $$\tan\phi_2 = r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta} = \frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta$$

or $$\phi_2 = \tfrac{1}{2}\theta \qquad (2)$$

From (1) and (2), the angle of intersection of the given curves

$$= \phi_1 \sim \phi_2 = \left(\tfrac{1}{2}\pi+\tfrac{1}{2}\theta\right) - \tfrac{1}{2}\theta = \tfrac{1}{2}\pi$$

which shows that the curves cut each other orthognally.

EXAMPLES

1. Find the angle ϕ for the curve $1/r = 1 + e\cos\theta$.

2. Find the angle between the radius vector and the tangent for the curve $2a/r = 1-\cos\theta$ at the point (r, θ). [*Rohilkhand* 2012]

3. Show that in the equiangular spiral $r = ae^{\theta\cot\alpha}$, the tangent is inclined at a constant angle to the radius vector.

4. Find the angle ϕ for the curve $a\theta = (r^2-a^2)^{1/2} - a\cos^{-1}(a/r)$.

5. Prove that the tangent at any point (r, θ) on the curve $r^2 = a^2\sin 2\theta$ makes an angle 3θ with the initial line.

6. Prove that the normal at any point (r, θ) on the curve $r^n = a^n\cos n\theta$ makes an angle $(n+1)\theta$ with the initial line. [*Kumaon* 2010]

7. Find the angle of intersection of the following curves :

   (i) $r = a\theta$, $r\theta = a$.

   (ii) $r = a\sin 2\theta$, $r = a\cos 2\theta$. [*Kanpur* 2005]

   (iii) $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$. [*Lucknow* 2009]

   (iv) $r = a\cos\theta$, $r = a(1-\cos\theta)$. [*Delhi* 2006]

   (v) $r = a/(1+\cos\theta)$, $r = b/(1-\cos\theta)$.

8. Prove that the spirals $r^n = a^n\cos n\theta$ and $r^n = b^n\sin n\theta$ intersect orthogonally.

[*Delhi* 03, *Bundelkhand* 03, *Guwahati* 05, *Kumaon* 08, 11]

9. Show that the curves $r^n = a^n\sec(n\theta+\alpha)$, $r^n = b^n\sec(n\theta+\beta)$ intersect at an angle which is independent of a and b. [*Lucknow* 2004]

10. Show that in the curve $r = n\theta$ the polar subnormal is constant and in the curve $r\theta = a$ the polar subtangent is constant.

11. Prove that the length of the perpendicular from the pole on the tangent to the ellipse $l/r = 1 + e\cos\theta$ is given by

$$\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right)$$

12. Prove that, in the parabola $2a/r = 1-\cos\theta$,

    (i) $\phi = \pi - \frac{1}{2}\theta$, (ii) $p = a\operatorname{cosec}\frac{1}{2}\theta$,

    (iii) the polar subtangent $= 2a\operatorname{cosec}\theta$, (iv) $p^2 = ar$.

[*Lucknow* 2006]

13. Prove that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is $r = f'(\theta - \frac{1}{2}\pi)$. Hence show that the locus of the extremity of the polar subnormal of the equiangular spiral $r = ae^{m\theta}$ is another equiangular spiral. [*Lucknow* 2011]

14. Show that the pedal equation

    (i) of the hyperbola $r^2\cos 2\theta = a^2$ is $pr = a^2$ [*Jabalpur* 2002]

    (ii) of the lemniscate $r^2 = a^2\cos 2\theta$ is $r^3 = a^2 p$.

(iii) of the Archimedian spiral $r = a\theta$ is $p^2 = r^4/(r^2 + a^2)$.

(iv) of the sine spiral $r^n = a^n \sin n\theta$ is $pa^n = r^{n+1}$.

15. Find the pedal equation of the curve $r = ae^{\theta \cot \alpha}$.

[*Lucknow* 2012]

16. Find the pedal equation of the curve $l/r = 1 + e\cos\theta$.

[*Bundelkhand* 2014]

17. Obtain the pedal equation of the curve $x = ae^{\theta}(\sin\theta - \cos\theta)$, $y = ae^{\theta}(\sin\theta + \cos\theta)$.

[*Lucknow* 2010]

18. Show that the pedal equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2 b^2}$.

[*Delhi* 04, *Calcutta* 05, *Kumaon* 13]

ANSWERS

1. $\tan^{-1}\{(1 + e\cos\theta)/e\sin\theta\}$. 2. $\pi - \frac{1}{2}\theta$.

4. $\cos^{-1}(a/r)$. 7. (i) $\frac{1}{2}\pi$ (ii) $\tan^{-1}(4/3)$.

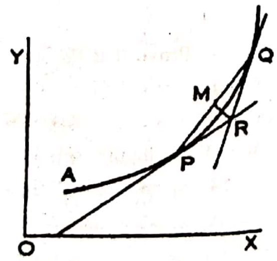
(iii) $\frac{1}{4}\pi$. (iv) $\frac{2}{3}\pi$. (v) $\frac{1}{2}\pi$.

15. $p = r\sin\alpha$. 16. $\dfrac{1}{p^2} = \dfrac{1}{l^2}\left(\dfrac{2l}{r} - 1 + e^2\right)$. 17. $p = \sqrt{2}r$.

~~8.9.~~ Differential coefficient of length of arc. Cartesian formula. Let the length of the arc *AP* of a curve, measured from a fixed point *A* on it be *s*. Then *s* evidently is some function, say $g(x)$, of x.

The problem is to find *ds/dx* when we know only the equation to the curve.

Let P be the point (x, y) and let $Q(x + \delta x, y + \delta y)$ be any neighbouring point on the curve. Let the tangents at P and Q meet in R and let RM be the perpendicular from R on PQ.



Let the length of the arc AQ be equal to $s + \delta s$, so that the arc $PQ = \delta s$.

We shall assume the following as an axiom.

$$\text{chord } PQ < \delta s < PR + RQ. \qquad (1)$$

Now $PR + RQ = PM \sec \angle MPR + MQ \sec \angle MQR$.

Thus $PR + RQ$

$$= PM(\sec \angle MPR - 1) + MQ(\sec \angle MQR - 1) + PM + MQ.$$

Substituting this value of $PR + RQ$ in (1) and dividing by δx, we get

$$\frac{PQ}{\delta x} < \frac{\delta s}{\delta x} < \frac{PM}{\delta x}(\sec \angle MPR - 1) + \frac{MQ}{h}(\sec \angle MQR - 1) + \frac{PQ}{\delta x} \qquad (2)$$

But $PQ^2 = (x + \delta x - x)^2 + (y + \delta y - y)^2 = (\delta x)^2 + (\delta y)^2$

$$\text{Therefore } \lim_{\delta x \to 0} \frac{PQ}{\delta x} = \lim_{\delta x \to 0} \frac{\sqrt{(\delta x)^2 + (\delta y)^2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \sqrt{1 + \frac{(\delta y)^2}{(\delta x)^2}} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad (3)$$

Also as $\delta x \to 0$, i.e. as the point $Q \to P$, PQ tends to the tangent PR, i.e. $\angle MPR \to 0$. Therefore $(\sec \angle MPR - 1) \to 0$. Moreover, $PM/\delta x$ does not tend to infinity, because $PQ/\delta x$ does not tend to infinity (see above).

$$\text{Thus} \quad \lim_{\delta x \to 0} \frac{PM}{\delta x}(\sec \angle MPR - 1) = 0.$$

Again, as the point $Q \to P$, the tangent at Q tends to the tangent at P. Therefore the exterior angle $R \to 0$. But, because $\angle MPR \to 0$, this means that $\angle MQR$ also $\to 0$. Thus, we have also

$$\lim_{\delta x \to 0} \frac{MQ}{\delta x}(\sec \angle MQR - 1) = 0.$$

Also, as is well known, when we take limits, an inequality might have to be replaced by an equality.

We know that $\dfrac{ds}{dx} = \lim_{\delta x \to 0} \dfrac{\delta s}{\delta x}$

Thus from (2) and (3) we get, upon taking the limits of the quantities involved,

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} \le \frac{ds}{dx} \le \sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

This shows that $\boxed{\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}.}$

8.91. Corollary. We have proved above that

$$\lim_{\delta x\to 0} PQ/\delta x = \lim_{\delta x\to 0} \delta s/\delta x$$

from which we can infer at once that $\lim_{\delta x\to 0} PQ/\delta s = 1$,

i.e. $\quad \boldsymbol{\lim_{\delta x\to 0}}$ **(chord PQ / arc PQ) = 1.**

8.92. Alternative Proof of the Formula for ds/dx. If we assume that

$$\lim_{\delta x\to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = 1 \qquad (1)$$

we can deduce the value of ds/dx more easily as follows :

By (1), $\quad \lim_{\delta x\to 0} \dfrac{\text{chord } PQ}{\delta x}\cdot\dfrac{\delta x}{\text{arc } PQ} = 1,$

i.e. $$\lim_{\delta x\to 0}\frac{\text{chord } PQ}{\delta x} = \lim_{\delta x\to 0}\frac{\text{arc } PQ}{\delta x}. \qquad (2)$$

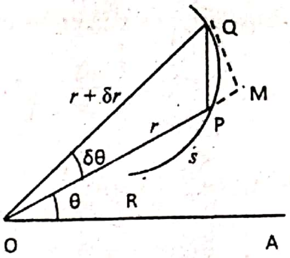
But the left-hand side of (2)

$$= \lim_{\delta x\to 0}\frac{\sqrt{(\delta x)^2+(\delta y)^2}}{\delta x} = \lim_{\delta x\to 0}\sqrt{1+\frac{(\delta y)^2}{(\delta x)^2}} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

and the right-hand side of (2) $= \lim_{\delta x\to 0}\dfrac{\delta s}{\delta x} = \dfrac{ds}{dx}$.

Hence $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

8.10. Differential coefficient of the length of arc. Polar formula. Let P be any point (r, θ) on the curve $r = f(\theta)$, and let Q, any neighbouring point on the curve, be $(r+\delta r,\ \theta+\delta\theta)$. Let s be the length of the arc up to P, measured from some fixed point R on the curve and let arc $RQ = s+\delta s$ so that arc $PQ = \delta s$. Draw QM perpendicular to OP (produced if necessary).



Now from ΔPQM,

$$(\text{chord } PQ)^2 = PM^2 + MQ^2 = (OM-OP)^2 + MQ^2$$
$$= [(r+\delta r)\cos\delta\theta - r]^2 + [(r+\delta r)\sin\delta\theta]^2$$
$$= [(r+\delta r)\cdot 1 - r]^2 + [(r+\delta r)\,\delta\theta]^2.$$

since as $Q \to P$, $\delta\theta \to 0$, so $\sin\delta\theta \to \delta\theta$ and $\cos\delta\theta \to 1$,

or $$(\text{chord } PQ)^2 = (\delta r)^2 + (r\,\delta\theta)^2 \qquad (1)$$

on neglecting $\delta r\,\delta\theta$ which is a very small quantity of the second order.

Therefore $$\frac{ds}{d\theta} = \lim_{\delta\theta\to 0}\frac{\delta s}{\delta\theta} = \lim_{\delta\theta\to 0}\frac{\text{arc } PQ}{\delta\theta}$$
$$= \lim_{\delta\theta\to 0}\frac{\text{chord } PQ}{\delta\theta} \qquad (2)$$

since in the limiting position as $Q \to P$, arc $PQ \to$ chord PQ.

Hence, from (1) and (2), we have

$$\frac{ds}{d\theta} = \lim_{\delta\theta\to 0}\frac{\text{chord } PQ}{\delta\theta} = \lim_{\delta\theta\to 0}\frac{\sqrt{(\delta r)^2 + (r\,\delta\theta)^2}}{\delta\theta}$$
$$= \lim_{\delta\theta\to 0}\sqrt{\left(\frac{\delta r}{\delta\theta}\right)^2 + r^2}.$$

Thus $$\boxed{\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.}$$

Similarly, we can prove that

$$\frac{ds}{dr}=\sqrt{1+\left(r\frac{dr}{d\theta}\right)^2}.$$

Alternatively, $\dfrac{ds}{dr}=\dfrac{ds}{d\theta}\dfrac{d\theta}{dr}=\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\cdot\dfrac{d\theta}{dr}$

$$=\sqrt{r^2\left(\frac{d\theta}{dr}\right)^2+\left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{dr}\right)^2}=\sqrt{\left(r\frac{d\theta}{dr}\right)^2+1}.$$

8.111. Other formulae.

(1) We know that $dy/dx=\tan\psi$, where ψ is the angle which the tangent makes with the x-axis. Hence

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\tan^2\psi}=\pm\sec\psi$$

Therefore $\qquad \dfrac{dx}{ds}=\cos\psi.$

if s be measured in such a way that dx/ds is positive, i.e. if x and s increase together.

(2) Again $\qquad \dfrac{dy}{ds}=\dfrac{dy}{dx}\cdot\dfrac{dx}{ds}=\tan\psi\ \cos\psi=\sin\psi.$

if x, y and s increase together.

(3) If x and y are given in terms of a parameter, say $x=f_1(t)$, $y=f_2(t)$, then

$$\frac{ds}{dt}=\frac{ds}{dx}\cdot\frac{dx}{dt}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot\frac{dx}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\right)^2\left(\frac{dx}{dt}\right)^2}$$

Hence $\qquad \dfrac{ds}{dt}=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}.$

(4) We have seen that $\tan\phi=r\,d\theta/dr$. Hence

$$\cos\phi=\frac{1}{\sec\phi}=\frac{1}{\sqrt{1+\tan^2\phi}}$$

$$=\frac{1}{\sqrt{1+r^2(d\theta/dr)^2}}=\frac{dr/d\theta}{\sqrt{(dr/d\theta)^2+r^2}}=\pm\frac{dr/d\theta}{ds/d\theta}$$

i.e. $$\boxed{\cos\phi=\frac{dr}{ds}}$$

s being measured in such a way that s and θ increase together, so that ϕ is acute when r increases with s (see § 8.5).

Also $\qquad \sin\phi=\tan\phi\cos\phi=\dfrac{r\,d\theta}{dr}\cdot\dfrac{dr}{ds}=r\dfrac{d\theta}{ds}. \qquad (1)$

(5) We know that $\quad p=r\sin\phi=r^2\dfrac{d\theta}{ds} \qquad$ [using (1)]

Hence $\qquad \dfrac{ds}{d\theta}=\dfrac{r^2}{p}. \qquad$ **[*Lucknow* 03, 07, 10]**

(6) Again $\quad \dfrac{dr}{ds}=\cos\phi=\sqrt{1-\sin^2\phi}=\sqrt{1-(p^2/r^2)}$

[since $p=r\sin\phi$]

Hence $\qquad \dfrac{ds}{dr}=\dfrac{r}{\sqrt{r^2-p^2}}.$

Ex. 1. Prove that $\quad \sin^2\phi\dfrac{d\phi}{d\theta}+r\dfrac{d^2r}{ds^2}=0. \qquad$ **[*Kanpur* 2010]**

Since $dr/ds=\cos\phi$, differentiating this w.r.t. s, we have

$$\frac{d^2r}{ds^2}=-\sin\phi\frac{d\phi}{ds}=-\sin\phi\frac{d\phi}{d\theta}\frac{d\theta}{ds}$$

or $\qquad \dfrac{d^2r}{ds^2}=-\sin\phi\dfrac{d\phi}{d\theta}\cdot\dfrac{\sin\phi}{r}=-\dfrac{\sin^2\phi}{r}\dfrac{d\phi}{d\theta}. \qquad$ [since $r\dfrac{d\theta}{ds}=\sin\phi$]

Hence $\qquad \sin^2\phi\dfrac{d\phi}{d\theta}+r\dfrac{d^2r}{ds^2}=0.$

Ex. 2. For the curve $\ r=ae^{\theta\cot\alpha}$, prove that $s/r=$ constant.

[*Lucknow* 2008]

Differentiating the given equation of the curve, we have

$$dr/d\theta=ae^{\theta\cot\alpha}(\cot\alpha)=r\cot\alpha.$$

This gives $r\dfrac{d\theta}{dr}=\tan\alpha.$

Then $$\frac{ds}{dr}=\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}=\sqrt{(1+\tan^2\alpha)}=\sec\alpha. \qquad (1)$$

Integrating (1) with respect to r, we get

$$s=r\sec\alpha+c, \qquad (2)$$

where c is the constant of integration.

It is given that s is measured from the origin so that $s=0$ when $r=0.$ Substitution in (2) gives

$$0=0\cdot\sec\alpha+c \quad \text{or} \quad c=0.$$

Hence $s=r\sec\alpha$ i.e. $s/r=\sec\alpha$, which is a constant.

EXAMPLES

1. Calculate ds/dx for the following curves :

   (i) $y=ax^2+bx+c.$ (ii) $y=\log\cos x.$

   (iii) $y=a\cosh(x/a).$

2. For the curve $y=a\log\sec(x/a)$, prove that

   $ds/dx=\sec(x/a).$ **[*Bundelkhand* 2013]**

3. For the parabola $y^2=4ax$, prove that $\dfrac{ds}{dx}=\sqrt{\dfrac{a+x}{x}}.$

   [*Lucknow* 2009]

4. Calculate ds/dt for the following curves :

   (i) $x=t^2,\ y=t-1.$ (iii) $x=2\sin t,\ y=\cos 2t.$

   (iii) $x=a\sec t,\ y=b\tan t.$

7. In the curve $y=a\log\sec(x/a)$, prove that

   $\dfrac{d^2x}{ds^2}=-\dfrac{1}{2a}\sin\dfrac{2x}{a}.$ **[*Lucknow* 2011]**

8. Find $ds/d\theta$ for the following curves:

   (i) $r=a(1+\cos\theta)$ (ii) $r^2=a^2\cos 2\theta$

   (iii) $r=a(\theta-1)$ (iv) $r=ae^{\theta\cot\alpha}$

9. Prove that for the curve $r\theta=a$, $\dfrac{ds}{dr}=\dfrac{\sqrt{r^2+a^2}}{r}.$

10. If $\dfrac{2a}{r}=1+\cos\theta$, prove that $\dfrac{ds}{d\phi}=\dfrac{2a}{\sin^3 d\phi}.$ **[*Lucknow* 2010]**

11. Show that $r\dfrac{ds}{dr}$ is constant for the curve **[*Kanpur* 2012]**

    $\theta=\cos^{-1}\dfrac{r}{k}-\dfrac{\sqrt{k^2-r^2}}{r}.$ [HINT : Put $r=k\cos t$]

ANSWERS

1. (i) $\sqrt{4a^2x^2+4abx+b^2+1}.$ (ii) $\sec x.$

(iii) $\cosh(x/a).$

4. (i) $\sqrt{1+4t^2}.$ (iii) $2\cos t\sqrt{1+4\sin^2 t}.$

(iii) $(a^2\sin^2 t+b^2)^{1/2}\sec^2 t.$

8. (i) $2a\cos\frac{1}{2}\theta.$ (ii) $a\sqrt{\sec 2\theta}.$

(iii) $a(\theta^2+1).$ (iv) $a\,\mathrm{cosec}\,\alpha\, e^{\theta\cot\alpha}.$

9

Asymptotes

9.1. Asymptotes. Some curves are limited in extent, for example the circle ; others extend to infinity, for example the parabola or the hyperbola. In the latter case, it is interesting to enquire what happens to the tangent when the point at which the tangent is drawn moves further and further away from the origin. There are three possibilities, corresponding to the three possibilities in the cases of a series, viz., that a series may be divergent, oscillatory or convergent. Thus it is possible that the tangent may go further and further away from the origin, or it may keep oscillating, or it may tend to a definite straight line. In the last case, the straight line to which the tangent tends is called the ***asymptote.*** The formal definition is as follows :

A straight line at a finite distance from the origin to which a tangent to a curve tends, as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve.

The tangent to the curve $y = f(x)$ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or

$$Y = \frac{dy}{dx} X + \left(y - x\frac{dy}{dx}\right). \qquad (1)$$

Excluding for the present asymptotes parallel to the y-axis, (1) shows that, as $x \to \infty$,

$$\frac{dy}{dx} \text{ and } y - x\frac{dy}{dx}$$

must both tend to finite limits, say m and c, in order that an asymptote might exist. If this condition is satisfied, the asymptote would be

$$y = mx + c.$$

9.11. The slope *m* is equal to the limit of y/x as x tends to infinity. We have seen that if $y = f(x)$ has an asymptote which is not parallel to the y-axis, then

$$\lim_{x\to\infty}\left(y - x\frac{dy}{dx}\right) \quad \text{is finite.}$$

It follows that

$$\lim_{x\to\infty}\frac{y - x\,dy/dx}{x} = 0,$$

i.e.

$$\lim_{x\to\infty}\frac{y}{x} - \lim_{x\to\infty}\frac{dy}{dx} = 0.$$

Therefore

$$\lim_{x\to\infty}\frac{y}{x} = \lim_{x\to\infty}\frac{dy}{dx} = m,$$

by the previous article; i.e., ***the slope m of an asymptote is equal to the limit of y/x as x tends to infinity.***

9.12. Branches of a curve. If y has two or more values for every value of x, it is generally more convenient to regard the curves corresponding to these distinct functions, not as different curves, but as *different branches* of one curve. In general, each branch has its own asymptote.

Thus if $y^2 - 2xy - 1 = 0$, then $y = x \pm \sqrt{x^2 + 1}$. So we say that the curves $y = x + \sqrt{x^2 + 1}$ and $y = x - \sqrt{x^2 + 1}$ are two branches of the curve $y^2 - 2xy - 1 = 0$, and each branch will have an asymptote.

9.2. The asymptotes of the general algebraic curve. Let the equation to the curve be

$$\begin{aligned} a_0 y^n + a_1 y^{n-1}x + a_2 y^{n-2}x^2 + \ldots + a_{n-1} y x^{n-1} + a_n x^n \\ + b_1 y^{n-1} x + b_2 y^{n-2} x + \ldots + b_{n-1} y x^{n-2} + b_n x^{n-1} \\ + c_2 y^{n-2} + \ldots + \ldots = 0. \end{aligned} \qquad (1)$$

or
$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+\ldots=0 \tag{2}$$

where $\phi_r(y/x)$ is an expression of the rth degree in y/x.

Dividing by x^n, this can be written as

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots=0. \tag{3}$$

Excluding the case of asymptotes parallel to the y-axis (i.e. excluding the case in which $\lim_{x\to\infty}(y/x)$ is equal to ∞), (3) gives, on taking limits as $x\to\infty$, the equation

$$\boldsymbol{\phi_n(m)=0} \tag{4}$$

where $m=\lim_{x\to\infty}(y/x)$.

This equation determines the m which occurs in the equation $y=mx+c$ of the asymptote (§ 9.11). [Since (4) is of degree n in m, there will be n values of m, corresponding to the the n branches of the curve (1). Some of the values of m may, however, be imaginary or coincident.]

Differentiating (3), we get

$$\left\{\phi'_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi'_{n-1}\left(\frac{y}{x}\right)+\ldots\right\}\frac{y'x-y}{x^2}$$
$$-\frac{1}{x^2}\phi_{n-1}\left(\frac{y}{x}\right)-\frac{2}{x^3}\phi_{n-2}\left(\frac{y}{x}\right)-\ldots=0.$$

Now multiply this out by x^2 and take limits as $x\to\infty$. Remembering that $\lim_{x\to\infty}(y'x-x)=-c$, by § 9.11, we get

$$\boldsymbol{c\,\phi'_n(m)+\phi_{n-1}(m)=0} \tag{5}$$

which determines one value of c for each value of m found from (4).

Hence the asymptotes are $y=mx+c$, where m is a root of (4) and the corresponding value of c is obtained from (5).

9.21. An Easy Rule for finding asymptotes. If we substitute $mx+c$ for y in (2) of the last article and expand, we get

$$x^n\left\{\phi_n(m)+\frac{c}{x}\phi'_n(m)+\frac{c^2}{2!x^2}\phi''_n(m)+\ldots\right\}$$
$$+x^{n-1}\left\{\phi_{n-1}(m)+\frac{c}{x}\phi'_{n-1}(m)+\ldots\right\}+\ldots=0. \tag{1}$$

If we now equate to zero the coefficients of the two highest powers of x, we get precisely the equations (4) and (5) of the last article. Hence we have the following rule for determining the asymptotes :

Substitute $mx+c$ for y in the equation of the curve and equate to zero the coefficients of the two highest powers of x. Determine m and c from these. If m_1, c_1; m_2, c_2; ... are the values of m and c thus obtained, the asymptotes are

$$y=m_1x+c_1;\ \ y=m_2x+c_2;\ldots.$$

Ex. Find the asymptotes of

$$y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0.$$

[*Kashi Vidyapeeth* **09**, ***Purvanchal*** **09**, ***Jabalpur*** **09**, ***Raipur*** **13**]

Putting $y=mx+c$ in the equation of the curve, we have

$$(mx+c)^3-x^2(mx+c)-2x(mx+c)^2+2x^3-7x(mx+c)$$
$$+3(mx+c)^2+2x^2+2x+2(mx+c)+1=0$$

or $\quad x^3(m^3-m-2m^2+2)+x^2(3m^2c-c-4mc-7m+3m^2+2)+\ldots=0$

Therefore m and c are given by

$$m^3-2m^2-m+2=0,$$

and
$$c(3m^2-4m-1)+3m^2-7m+2=0.$$

The first equation gives $(m-1)(m+1)(m-2)=0.$

i.e. $\quad m=1,-1$ or $2.$

The second equation gives $\quad c=-\dfrac{3m^2-7m+2}{3m^2-4m-1}.$

Therefore, when $m=1$, $c=-1$; when $m=-1$, $c=-2$; and when $m=2$, $c=0$.

Hence the asymptotes are $y=x-1$, $y=-x-2$ and $y=2x$.

9.22. Shorter Method. We notice that $\phi_n(m)$ can be obtained at once by putting $x=1$ and $y=m$ in the highest degree terms of the equation to the curve. Similarly $\phi_{n-1}(m)$ can be obtained by putting $x=1$ and $y=m$ in the $(n-1)$th degree terms. Hence we get the asymptotes more quickly as follows:

In the highest degree terms put $x=1$ and $y=m$, thus getting $\phi_n(m)$. Equate this to zero and solve for m. Let the roots be $m_1, m_2, \ldots m_n$. Next form $\phi_{n-1}(m)$ in a similar way from the terms of degree $n-1$. Then the values of c, say $c_1, c_2, \ldots c_n$ are found by substituting $m_1, m_2, \ldots m_n$ in turn in the formula

$$c=-\frac{\phi_{n-1}(m)}{\phi_n'(m)}.$$

The asymptotes then are $y=m_1x+c_1$, $y=m_2x+c_2, \ldots, y=m_nx+c_n$.

Ex. 1. Solve the example of the last article by this method.

The equation of the given curve is

$$y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0.$$

Putting $x=1$ and $y=m$ in the 3rd degree terms, and equating to zero, we get

$$m^3-2m^2-m+2=0 \qquad (1)$$

i.e. $(m-1)(m+1)(m-2)=0$, whence $m=1, -1$, or 2.

Also, from (1), $\phi_3'(m)=3m^2-4m-1$.

Again, putting $x=1$ and $y=m$ in the 2rd degree terms, we get

$$\phi_2(m)=3m^2-7m+2.$$

Then $$c=-\frac{\phi_2(m)}{\phi'_3(m)}=-\frac{3m^2-7m+2}{3m^2-4m-1}=-1, -2, 0$$

by substituting $m=1, -1$, or 2 respectively.

Therefore, the asymptotes are $y=x-1$, $y=-x-2$ and $y=2x$.

Ex. 2. Find all the asymptotes of the curves

$$x^3+2x^2y-xy^2-2y^3+xy-y^2=1. \qquad (1)$$

[*Kumaon* 04, *Patna* 03, 06, *Bilaspur* 07, 09, *Kashi Vidyapeeth* 11, *Bundelkhand* 06, 13, *Raipur* 14]

Putting $x=1$, $y=m$, in the 3rd and 2rd degree terms of (1), we get

$$\phi_3(m)=1+2m-m^2-2m^3, \qquad \phi_2(m)=m-m^2.$$

Then the slopes of the asymptotes of (1) will be given by

$$\phi_3(m)=0 \quad \text{i.e.} \quad 1+2m-m^2-2m^3=0.$$

or $$(1-m)(1+m)(1+2m)=0 \quad \text{whence} \quad m=1, -1, -1/2.$$

Now $\phi_3'(m)=2-2m-6m^2=2(1-m-3m^2)$.

Then $$c=-\frac{\phi_2(m)}{\phi'_3(m)}=\frac{m^2-m}{2(1-m-3m^2)}=0, -1, 1/2$$

for $m=1, -1, -1/2$ respectively. Hence the required asymptotes are

$$y=x,\ y+x+1=0 \text{ and } y=-\tfrac{1}{2}x+\tfrac{1}{2} \text{ or } 2y+x=1.$$

NOTE. If in § 9.4, x tends to $-\infty$, instead of to $+\infty$, the equations for finding m and c remain unaltered. Hence the same asymptotes will be obtained whether $x\to\infty$ or $x\to-\infty$.

EXAMPLES

Find the asymptotes of the following curves :

1. $x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0.$
2. $x^3+2x^2y-xy^2-2y^3+3xy+3y^2+x+1=0.$
3. $2x^3+3x^2y-3xy^2-2y^3+3x^2-3y^2+y=3.$
4. $4x^3-x^2y-4xy^2+y^3+3x^2+2xy-y^2=7.$ **[*Jabalpur* 2009]**
5. $y^3-6xy^2+11x^2y-6x^3+y+x=0.$
6. $y^3-x^2y+2y^2+4y+x=0.$ **[*Agra* 2006]**
7. $3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0.$ **[*Lucknow* 07, *Avadh* 07]**

ANSWERS

1. $x+y=1,\ x-y+1=0,\ x+2y=0.$
2. $2y+x=1,\ y=x+1,\ y+x=0.$
3. $y=x,\ y+2x+1=0, 2y+x+1=0.$
4. $y=-x,\ y=x+\frac{2}{3},\ y=4x+\frac{1}{3}.$

5. $y=x,\ y=2x,\ y=3x$.
6. $y=0,\ x-y=1,\ x+y+1=0$.
7. $y-x+7/6=0,\ y-3x+3/2=0,\ 2y+x+5/3=0;$
   $106y-381x+105=0$.

9.3. Asymptotes might not exist. If one or more values of m, found from $\phi_n(m)=0$, make $\phi'_n(m)$ zero, but do not make $\phi_{n-1}(m)$ zero, the equation for determining the corresponding value of c becomes

$$0.c+\phi_{n-1}(m)=0.$$

This means that the equation in c is $Fc+G=0$, where

$$\lim_{x\to\infty} F=0 \text{ and } \lim_{x\to\infty} G=\phi_{n-1}(m).$$

Hence $\lim_{x\to\infty} c=+\infty$ or $-\infty$, and this corresponds to the case when the tangent goes further and further away as $x\to\infty$. Thus the curve has no asymptotes.

Ex. Find the asymptotes of the curve $y^2=x$.

Putting $y=mx+c$, we get $(mx+c)^2-x=0$.

Equating to zero the coefficients of x^2 and x, we have

$$m^2=0, \text{ and } 2mc-1=0.$$

The first gives $m=0$. Then the second reduces to $-1=0$, which is impossible. Hence $y^2=x$ has no asymptotes.

9.4. Two Parallel Asymptotes. If any two real equal values of m found from $\phi_n(m)=0$ make $\phi'_n(m)=0$ and also $\phi_{n-1}(m)=0$, then the curve will have two parallel asymptotes. Hence the equation from which c is usually determined degenerates into the identity $0\cdot c+0=0$. To determine c, we have now the equation

$$\boxed{\tfrac{1}{2}c^2\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0,} \qquad (1)$$

which follows from equation (3) of § 9.2, on differentiating it twice, multiplying out by a suitable power of x, and taking limits as $x\to\infty$.

Hence from the quadratic equation (1) in c, we get two values of c corresponding to these repeated values of m. That is, we shall get a pair of parallel asymptotes.

Reference to equation (1) of § 9.21 shows that the *coefficient of* x^{n-2} in that equation *is the same as the left-hand side of* (1) *above*.

Hence the rule for finding asymptotes can be extended as follows: ***Substitute mx + c for y in the equation of the curve and equate to zero the coefficient of the highest power of x. If any value of m derived from this equation makes the coefficient of x^{n-1} identically zero, then the corresponding value of c should be determined from the equation obtained by putting the coefficient of x^{n-2} equal to zero.***

Ex. 1. Find the asymptotes of $y^3+x^2y+2xy^2-y+1=0$.

[***Indore*** **03,** ***Delhi*** **08,** ***Bilaspur*** **08,** ***Rohilkhand*** **09,** ***Lucknow*** **11**]

Putting $y=mx+c$ in the above equation, we have

$$(mx+c)^3+x^2(mx+c)+2x(mx+c)^2-(mx+c)+1=0$$

or

$$x^3(m^3+m+2m^2)+x^2(3m^2c+c+4mc)$$
$$+x(3mc^2+2c^2-m)+\ldots=0.$$

Equating to zero, the coefficients of x^3 and x^2, we get

$$m^3+2m^2+m=0 \quad \text{and} \quad (3m^2+4m+1)\ c=0.$$

[These could also have been written down at once by § 9.22]

The first equation gives $m=0,-1,-1$.

The second equation then gives $c=0$ when $m=0$. When $m=-1$, however, the second equation reduces to the identity $0.c=0$. [We could have anticipated this, because -1 is a repeated root of the first equation.]

In such cases, equating, therefore, the coefficient of the next lower power of x to zero [or by finding $\phi''_n(m)$ and $\phi_{n-2}(m)$], we have

$(3m+2)c^2-m=0$, which gives $c=\pm 1$ after putting $m=-1$.

Hence the asymptotes are $y=0$, $y=-x+1$ and $y=-x-1$.

NOTE. *The student must guard against concluding from equations like* $(3m^2+4m+1)c=0$ *that* $c=0$. *This inference would be correct only when the coefficient of* c *is not zero.*

The student should remember that such a case will arise whenever two or more of the roots of the equation for finding m are equal.

Ex. 2. Find the asymptotes of the cubic curve

$$y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x-1=0$$

[*Rewa* **05,** *Jabalpur* **04, 08,** *Kashi Vidyapeeth* **10]**

Putting $y=mx+c$ in the equation, we obtain

$$(mx+c)^3-5x(mx+c)^2+8x^2(mx+c)-4x^3-3(mx+c)^2$$
$$+9x(mx+c)-6x^2+2(mx+c)-2x=1.$$

or
$$x^3(m^3-5m^2+8m-4)+x^2(3m^2c-10mc$$
$$+8c-3m^2+9m-6)+\ldots..=0. \qquad (1)$$

Equating to zero, the coefficients of x^3 and x^2, we get

$$m^3-5m^2+8m-4=0,$$

and
$$(3m^2-10m+8)c-3m^2+9m-6=0.$$

The first equation gives $m=1, 2, 2$.

If $m=1$, then the second equation gives $c=0$. Hence, the corresponding asymptote is $y=x$.

If $m=2$, the second equation reduces to the identity $0.c+0=0$. Therefore, to find c, equate the next lower coefficient (of x) to zero in equation (1). This gives

$$3mc^2-5c^2-6mc+9c+2m-2=0.$$

Putting $m=2$ in this, we get

$$c^2-3c+2=0 \quad \text{or} \quad c=1 \text{ or } 2.$$

So, for $m=2$, there are two parallel asymptotes

$$y=2x+1 \quad \text{and} \quad y=2x+2.$$

Hence the asymptotes are $y=x$, $y=2x+1$ and $y=2x+2$.

EXAMPLES

Find the asymptotes of the following curves :

1. $x^3+x^2y-xy^2-y^3-3x-y-1=0$. [*Meerut* **04,** *Gorakhpur* **06]**
2. $x^3+x^2y-xy^2-y^3+x^2-y^2-2=0$.
3. $4x^3-3xy^2-y^3+2x^2-xy-y^2=1$.

   [*Avadh* **06,** *Agra* **09,** *Magadh* **10]**
4. $x^3+3x^2y-4y^3-x+y+3=0$.

   [*Delhi* **04,** *Avadh* **05, 09,** *Rohilkhand* **10,** *Agra* **11]**

5. $x^3-5x^2y+8xy^2-4y^3+x^2-3xy+2y^2-1=0$.

   [*Raipur* **11,** *Patna* **2011]**
6. $2x^3-x^2y-2xy^2+y^3-4x^2+8xy-4x+1=0$.

   [*G.N.D.U. Amritsar* **04,** *Delhi* **03, 06,** *Raipur* **09]**

ANSWERS

1. $y=x,\ y=-x-1,\ y=-x+1$.
2. $y=x,\ y=-x,\ y=-x-1$.
3. $y=x,\ y=-2x,\ y=-2x-1$.
4. $y=x,\ 2y=-x+1,\ 2y=-x-1$.
5. $y=x,\ 2y=x,\ 2y=x+1$.
6. $x+y=2,\ x-y+2=0,\ 2x-y=4$.

9.41. More than two Parallel Asymptotes. If three values of m derived from $\phi_n(m)=0$ are equal, then the coefficient of x^{n-2} equated to zero becomes the identity

$$0.c^2+0.c+0=0$$

So c must be determined from the equation obtained by differentiating the equation of the curve thrice, multiplying out by a suitable power of x and taking limits as $x \leftarrow \infty$.

The same equation can be obtained more quickly, however, by equating to zero the coefficient of the next highest power of x in the equation obtained on writing $mx+c$ for y in the equation of the curve (the law of formation of the coefficients in Taylor's series shows at once why the two equations are the same).

If this equation also fails to determine c, then the coefficient of the next highest power of x must be equated to zero, and so on. These cases occur when the equation $\phi_n(m)=0$ has three, four or more equal roots.

9.42. Asymptotes Parallel to the x-axis. While asymptotes parallel to the x-axis, for which $m=0$, would get determined when we substitute $y=mx+c$ and equate to zero the coefficients of x^n and x^{n-1}, it is usual to determine these separately first, as the method of finding them is extremely simple.

The general equation of the curve of degree n (eqn. (1) of § 9.2), arranged according to powers of x, is

$$a_n x^n + (a_{n-1} y + b_n) x^{n-1} + (a_{n-2} y^2 + b_{n-1} y + c_n) x^{n-2} + \ldots = 0. \qquad (1)$$

Putting $y = mx + c$, and equating to zero the coefficient of x^n (or at once by § 9.22), we get for determining m the equation

$$a_n + a_{n-1} m + a_{n-2} m^2 + \ldots = 0. \qquad (2)$$

This shows that $m = 0$ will be a root only if $a_n = 0$.

Suppose then that $a_n = 0$, and consider the root $m = 0$ of (2). The corresponding asymptote is

$$y = c \qquad (3)$$

where c is determined by the equation obtained on substituting $y = 0 . x + c$ (i.e. $y = c$) in (1) and equating to zero the coefficient of x^{n-1}. Hence c in (3) must be determined by

$$a_{n-1} c + b_n = 0. \qquad (4)$$

Now, to substitute the value of c from (4) in (3) is the same as eliminating c between (4) and (3). Hence the asymptote is

$$a_{n-1} y + b_n = 0, \qquad (5)$$

which could have been obtained by equating to zero the coefficient of x^{n-1} in (1).

If a^{n-1} and b_n are both zero, so that (4) becomes an identity and there are no terms involving x^{n-1} in the equation to the curve, arguments similar to those of parallel asymptotes will apply. We shall find that in every case ***the asymptote or asymptotes parallel to the axis of x can be obtained by equating to zero the coefficient of the highest power of x, provided this is not merely a constant.***

Ex. 1. Find the asymptote, parallel to the axis of x, of the curve $y^3 + x^2 y + 2xy^2 - y + 1 = 0$. **[*Bilaspur* 08, *Delhi* 08]**

Here the highest power of x is x^2 and its coefficient is y. So the required asymptote is $y = 0$. (Cf. the Ex. of § 9.4)

Ex. 2. Find the asymptotes, parallel to the axis of x, of the curve

$$y^4 + x^2 y^2 + 2xy^2 - 4x^2 - y + 1 = 0.$$

Here the coefficient of the highest power of x is $y^2 - 4$.

So the required asymptotes are $y = \pm 2$.

9.43. Asymptotes parallel to the y-axis. As the form $x = c$ is not included in $y = mx + c$. the preceding methods will fail to determine asymptotes of this form. But they can be easily found from the rule given below, whose truth becomes apparent when we consider what would be the result of interchanging the axes of x and y.

The asymptotes parallel to the axis of y are obtained by equating to zero the coefficient of the highest power of y, provided this is not merely a constant.

Ex. 1. Find those asymptotes of the curve

$$x^4 + x^2 y^2 - a^2 (a^2 + y^2) = 0$$

which are parallel to the y-axis.

➔ The coefficient of y^2 equated to zero gives $x^2 - a^2 = 0$. Hence the required asymptotes are $x = a, x = -a$.

Ex. 2. Show that the asymptotes of the curve

$$x^2 y^2 - a^2 (x^2 + y^2) - a^3 (x + y) + a^4 = 0$$

form a square, through two of whose angular points the curve passes. **[*Garhwal* 08, *Bilaspur* 10]**

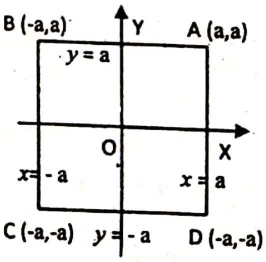
➔ The equation of the given curve can be written as

$$x^2 (y^2 - a^2) - a^2 y^2 - a^3 (x + y) + a^4 = 0. \qquad (1)$$

Equating to zero, the coefficient of highest power of x in (1), i.e. $y^2 - a^2 = 0$ or $y = \pm a$. gives these asymptotes parallel to the x-axis.

Again, let us write the equation of the curve as

$$y^2 (x^2 - a^2) - a^2 x^2 - a^3 (x + y) + a^4 = 0, \qquad \ldots\ldots (2)$$



Then the asymptotes parallel to the y-axis, obtained by equating to zero the coefficient of highest power of y in (2), are given by

$$x^2-a^2=0 \quad \text{or} \quad x=\pm a.$$

These are all the asymptotes of the curve whose equation being of the 4th degree, it can have no more than four asymptotes. (Cf. § 9.61.)

Thus, it is obvious that the asymptotes $y=\pm a$, $x=\pm a$, form a square $ABCD$ whose angular points are $A\,(a,a)$, $B\,(-a,a)$, $C\,(-a,-a)$ and $D\,(a,-a)$.

Since the points $B\,(-a,a)$ and $D\,(a,-a)$ satisfy the equation of the curve, hence the given curve passes through these two angular points.

EXAMPLES

Find the asymptotes of the following curves :

1. $x^3+3xy^2+y^2+2x+y=0.$
2. $a^2/x^2+b^2/y^2=1.$ [*Agra* 2009]
3. $y^2(a^2-x^2)=x^4.$ [Meerut 2010]
4. $x^2y^3+x^3y^2=x^3+y^3.$ [*Rohilkhand* 07, *Kumaon* 10]
5. $xy^3+x^3y=a^4.$ [*Agra* 2006]
6. $(x^3+a^3)y=bx^3.$ [*Rohilkhand* 2011]
7. Prove that the four asymptotes of the curve $x^2y^2-x^2y-xy^2+x+y+1=0$ form a square. [*Agra* 2008]

ANSWERS

1. $3x+1=0.$
2. $y=\pm a,\ x=\pm b.$
3. $x=\pm a.$
4. $y=\pm 1,\ x=\pm 1.$
5. $y=0,\ x=0.$
6. $y=b,\ x=-a.$

9.5. The asymptotes of the curve $y = mx + c + A/x + B/x^2 + \ldots$.

Let the equation to a curve be written as

$$y=mx+c+\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\ldots \qquad (1)$$

where the series $A/x+B/x^2+\ldots$ is convergent for sufficiently large values of x.

Differentiating, we have

$$\frac{dy}{dx}=m-\frac{A}{x^2}-\frac{2B}{x^3}-\ldots.$$

The tangent at (x, y) is, therefore,

$$Y-y=\left(m-\frac{A}{x^2}-\frac{2B}{x^3}-\ldots\right)(X-x)$$

or
$$Y=\left(m-\frac{A}{x^2}-\frac{2B}{x^3}-\ldots\right)X+c+\frac{2A}{x}+\frac{3B}{x^2}+\ldots \qquad (2)$$

substituting for y from (1). Let x now tend to infinity. The equation (2) then tends to the equation

$$Y=mX+c.$$

We have proved, therefore, that ***the asymptote of the curve*** $\boldsymbol{y=mx+c+\frac{A}{x}+\frac{B}{x^2}+\ldots}$ **is**

$$\boldsymbol{y=mx+c.}$$

This method is sometimes useful.

Ex. Find the asymptotes of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1.$

Here $$\frac{y}{b}=\pm\sqrt{\frac{x^2}{a^2}-1}=\pm\frac{x}{a}\sqrt{1-\frac{a^2}{x^2}}=\pm\frac{x}{a}\left(1-\frac{1}{2}\frac{a^2}{x^2}-\frac{1}{8}\frac{a^4}{x^4}+\ldots\right).$$

Hence the asymptotes are $y=bx/a$ and $y=-bx/a.$

9.51. Position of the curve with respect to the asymptote. If we can put the equation of a given curve in the form $y=mx+c+Ax^{-1}+\ldots$, we can get useful information regarding the position of the curve with respect to the asymptote. The following cases arise :

Case I. $A\neq 0.$

Let y_1 be the ordinate of the curve and y_2 that of the asymptote when the absicca is x_1. Then

$$y_1=mx_1+c+A\,x_1^{-1}+B\,x_1^{-2}+\ldots \quad \text{and} \quad y_2=mx_1+c.$$

Now by taking x_1 sufficiently large, we can (in general) make $Bx_1^{-1}+Cx_1^{-2}+\ldots$ numerically as small as we please. Let x_1 be so large that this expression is numerically less than A.

Then $$A + Bx_1^{-1} + Cx_1^{-2} + \ldots$$
has the same sign as A; i.e., it is positive if A is positive and negative if A is negative.

Now $$y_1 - y_2 = x_1^{-1}(A + Bx_1^{-1} + \ldots)$$

Therefore, if x_1 and A are of the same sign, then $y_1 > y_2$, i.e. the curve is above the asymptote. On the other hand, if x_1 and A are of opposite signs, the curve lies below the asymptote.

We notice that the curve lies on opposite sides of the asymptote at the opposite ends.

Case II. $A = 0$.

Here the curve lies on the same side of the asymptote at opposite ends - above it if B is positive, and below it if B is negative.

Case III. If B is also zero, but not C, arguments similar to those of Case I would apply; and so on.

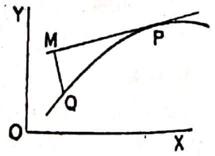
Such considerations would be found very helpful in curve tracing.

9.52. The Curve approaches the asymptote.

The distance of a point on any branch of a curve from the corresponding asymptote tends to zero as the distance of the point from the origin tends to infinity.

Let the distance of a point Q on the curve from the tangent to the curve at P be QM. Then we know that QM tends to zero as Q tends to P.

Similarly, when a point Q on a curve tends to infinity, its distance from the corresponding asymptote (which may be roughly regarded as a tangent with its point of contact a great distance away) tends to zero.



9.6. Alternative method of finding asymptotes of algebraic curves.

The following method of finding asymptotes is often more convenient than that of § 9.21. or § 9.22.

Suppose that the equation to the curve is of degree n, as in § 9.2.

Case. I. Non-repeated factor in the *n*th degree terms. Let $y - ax$ be a non-repeated factor of the nth degree terms of the equation to the curve. Then the equation to the curve can be written as
$$(y - ax)F_{n-1} + P_{n-1} = 0. \qquad (1)$$
where F_{n-1} contains only terms of degree $n-1$, and P_{n-1} contains terms of various degrees, none of which is of a degree higher than $n-1$.

Writing (1) as $$y - ax + P_{n-1}/F_{n-1} = 0$$
and taking the limit of P_{n-1}/F_{n-1} as $x \to \infty$, we shall get the equation of the asymptote, since the curve approaches the asymptote when $x \to \infty$.

This limit can be easily found if we remember that
$$\lim_{x\to\infty}(y/x) = a \qquad \text{(by § 9.11)}$$
Hence the asymptote corresponding to the factor $y - ax$ is
$$y - ax + \lim_{x\to\infty,\, y/x\to a}(P_{n-1}/F_{n-1}) = 0.$$

Ex. 1. Solve by this method the example of § 9.21.

We have
$$y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0.$$

Factorising the highest (third) degree terms, the equation to the curve can be written as
$$(y-x)(y+x)(y-2x) - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0.$$

Hence one asymptote is
$$y - x = \lim_{x\to\infty,\, y/x\to 1}\frac{7xy - 3y^2 - 2x^2 + \text{terms of lower degree}}{(y+x)(y-2x)}$$
$$= \lim_{x\to\infty,\, y/x\to 1}\frac{7(y/x) - 3(y/x)^2 - 2 + \text{terms which} \to 0}{(y/x+1)(y/x-2)}$$
$$= \frac{7-3-2}{(1+1)(1-2)} = -1, \text{ i.e. one asymptote is } y - x = -1.$$

Similarly, a second asymptote is

$$y+x=\lim_{x\to\infty,\, y/x\to 1}\frac{7(y/x)-3(y/x)^2-2}{(y/x-1)(y/x-2)}=-2$$

The third asymptote $y=2x$ can also be found in the same way

Ex. 2. Find the asymptotes of the Folium

$$x^3+y^3-3axy=0.$$

[*Himachal* 05, 06, 07, *Avadh* 08, *Bundelkhand* 09, *Agra* 05, 10, *Patna* 10, *Magadh* 11, *Lucknow* 05, 12, *Kashi Vidyapeeth* 12, *Kanpur* 12]

Factorising the third degree terms, the equation to the Folium can be written as

$$(x+y)(x^2-xy+y^2)-3axy=0$$

Since $(y+x)$ is the only real non - repeated factor of the highest degree terms of the Folium, the curve has only one real asymptote parallel to $y+x=0$ with a slope of -1 i.e. $y/x\to -1$.

Hence the asymptote is

$$y+x=\lim_{x\to\infty,\, y/x\to -1}\frac{3axy}{x^2-xy+y^2}$$

$$=\lim_{x\to\infty,\, y/x\to -1}\frac{3a(y/x)}{1-(y/x)+(y/x)^2}=\frac{3a(-1)}{1+1+1}=-a.$$

Therefore, the only asymptote of the Folium is $y+x+a=0.$

Case. II. Repeated factor in the nth degree terms but no factor in the $(n-1)$th degree terms. Let the terms of the nth degree in the equation to the curve contain $(y-ax)^2$ as a factor, and suppose $y-ax$ is not a factor of the $(n-1)$th degree terms. Then, proceeding as in Case 1, we find that $\lim_{x\to\infty,\, y/x\to a}(y-ax)^2$ is either $+\infty$, or $-\infty$. Hence there is no asymptote in this case (Cf. § 9.3).

Case. III. Repeated factor in the nth degree terms and a factor in the $(n-1)$th degree terms. Let the equation to the curve be of the form

$$(y-ax)^2F_{n-2}+(y-ax)G_{n-2}+P_{n-2}=0$$

where F_{n-2} and G_{n-2} contain only terms of degree $n-2$, and P_{n-2} contains terms none of which is of a degree higher than $n-2$.

Dividing by F_{n-2} and taking limits as $x\to\infty$ and $y/x\to a$, we get an equation of the form

$$(y-ax)^2+B(y-ax)+C=0,$$

which, on solving as a quadratic for $y-ax$, gives us two asymptotes of the form

$$y-ax=c_1 \text{ and } y-ax=c_2.$$

Case. IV. We can proceed in the same way if the nth degree terms contain $(y-ax)^3$, or a higher power of $y-ax$, as a factor.

IMPORTANT. If the equation to the curve is of the form

$$(ax+by+c)P_{n-1}+Q_{n-1}=0 \qquad (2)$$

where P_{n-1} and Q_{n-1} contain terms none of which is of a degree higher than $n-1$, and P_{n-1} contains at least one term of degree $n-1$ (to ensure that the equation of the curve is of degree n), a little consideration (or working out a few examples) will show that the asymptote corresponding to the factor $ax+by+c$ is

$$ax+by+c+\lim_{x\to\infty,\, y/x\to a/b}(Q_{n-1}/P_{n-1})=0$$

and that a similar modification can be made in the other cases. Thus we need not transform an equation of the form (2) into an equation of the form (1) [Case I] for finding out the asymptotes.

Ex. 1. Solve by this method the first example of § 9.4.

We have $$y^3+x^2y+2xy^2-y+1=0.$$

By factorising the terms of the third degree, the equation to the curve can be written as $$y(y+x)^2-y+1=0.$$ **[*Kanpur* 2008]**

The asymptotes corresponding to the factor $(y+x)^2$ are

$$(y+x)^2 + \lim_{x\to\infty,\, y/x\to -1} \frac{-y+1}{y} = 0$$

i.e. $$(y+x)^2 = -\lim_{x\to\infty,\, y/x\to -1} \frac{-y/x+1/x}{y/x} = -\frac{1+0}{-1} = 1;$$

i.e. $$y+x=1 \quad \text{and} \quad y+x=-1.$$

The third asymptote is $y=0$ by § 9.42.

Ex. 2. Find the asymptotes of

$$(x-y-1)^2(x^2+y^2+2)+6(x-y-1)(xy+7)-8x^2-2x-1=0.$$

Dividing by the coefficient of $(x-y-1)^2$, and taking limits we see that the asymptotes parallel to $x-y-1=0$ are

$$(x-y-1)^2+6(x-y-1)\lim_{x\to\infty,\, y/x\to 1}\frac{xy+7}{x^2+y^2+2}$$

$$+\lim_{x\to\infty,\, y/x\to 1}\frac{-8x^2-2x-1}{x^2+y^2+2}=0.$$

or $$(x-y-1)^2+6(x-y-1)\lim_{y/x\to 1}\frac{y/x}{1+(y/x^2)}$$

$$+\lim_{y/x\to 1}\frac{-8}{1+(y/x)^2}=0.$$

or $$(x-y-1)^2+3(x-y-1)-4=0$$

or $$x-y-1=\frac{-3\pm\sqrt{9+16}}{2}=1,-4.$$

Hence two of the asymptotes are $x-y-2=0$ and $x-y+3=0$. The other two asymptotes are imaginary since the remaining linear factors of the fourth degree terms in the equation to the curve are imaginary.

9.61. Total number of asymptotes. As the equation for determining m, viz. $\phi_n(m)=0$, is of degree n, and so has n roots, and as one value of m gives, in general, one value of c (§ 9.2), it is evident *that a curve of degree n has, in general, n asymptotes.*

(i) If some of the roots of $\phi_n(m)=0$, are imaginary, the corresponding asymptotes are said to be imaginary. Thus the circle $x^2+y^2=a^2$ has imaginary asymptotes.

(ii) There might be no asymptote corresponding to even a real root (§ 9.3). Thus the parabola $y^2=4ax$ has no asymptotes, even though the roots of $\phi_n(m)=0$ are real.

(iii) There can never be more than n asymptotes, because when the equation for determining c is a quadratic (§ 9.4), $\phi'_n(m)=0$, and so $\phi_n(m)=0$, has a double root (by the theory of equations). Therefore the two values of c correspond to the two equal roots, and there would be at the most $n-2$ other asymptotes corresponding to the remaining $n-2$ roots. If the equation for determining c is a cubic, it can be easily shown that $\phi_n(m)=0$, has three equal roots; and so on.

Hence we see that ***a curve of degree n can never have more than n asymptotes.***

Ex. Find all the asymptotes of $a^2/x^2-b^2/y^2=1.$

[***Garhwal*** **07,** ***Agra*** **07, 08,** ***Lucknow*** **09,** ***Bundelkhand*** **11**]

We can write the equation of the curve as

$$x^2y^2+b^2x^2-a^2y^2=0. \tag{1}$$

Equating to zero, the coefficient of highest power of x in (1) gives the asymptotes parallel to the x-axis as

$$y^2+b^2=0 \quad \text{i.e.} \quad y=\pm ib \quad \text{which are imaginary.}$$

Again equating to zero, the coefficient of highest power of y in (1) gives the asymptotes parallel to the y-axis as

$$x^2-a^2=0 \quad \text{i.e.} \quad x=\pm a.$$

As the equation to the curve is of the 4th degree, there cannot be more than four asymptotes. Hence all the asymptotes of the given curve are the real asymptotes $x=\pm a.$

9.62. Asymptotes by Inspection. *If the equation to a curve is of the form $F_n+P=0$, where F_n is of degree n (i.e. contains terms of degree n and may also contain terms of lower degree), and P is of degree $n-2$, or lower, and if $F_n=0$ can be broken up into n linear factors which represent n straight lines, no two of which are parallel or coincident, then all the asymptotes are given by $F_n=0$.*

The supposition that $F_n = 0$ breaks up into factors which represent straight lines, no two of which are parallel, implies that $\phi_n(m) = 0$ has no repeated roots. Hence the values of c do not depend on terms of degree lower than $n-1$ (see § 9.4).

So the asymptotes of $F_n + P = 0$ would be the same as those of $F_n = 0$. But $F_n = 0$ is a system of n straight lines, each one of which is its own asymptote (as is obvious from the definitions of asymptote and tangent). Hence the proposition must be true.

This enables us to write down the asymptotes by mere inspection in some cases. The above proposition will be true even if some of the linear factors of F_n are imaginary, i.e. involve $\sqrt{-1}$.

Ex. 1. Find the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

[*Gorakhpur* 2005]

By the above proposition, the equation of the hyperbola is in the form $F_2 + 0 = 0$ so that $F_2 = 0$ gives

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{i.e.} \quad \frac{x}{a} = \pm\frac{y}{b}$$

as the required asymptotes.

Ex. 2. Find the asymptotes of the curve

$$(x-y)(x+y)(x+2y-1) = 3x + 4y + 5.$$ [*Raipur* 2012]

The equation of the curve can be written as

$$(x-y)(x+y)(x+2y-1) - (3x+4y+5) = 0.$$

This is of the form $F_3 + F_1 = 0$ where no two of the lines $x - y = 0$, $x + y = 0$ and $x + 2y + 1 = 0$ are parallel.

Hence, by inspection, the asymptotes of the curve are given by $F_3 = 0$ i.e. $x - y = 0$, $x + y = 0$ and $x + 2y + 1 = 0$.

EXAMPLES

Find all the asymptotes of the following curves :

1. $y^3 = x^3 + ax^2$
2. $y^2(x^2 - a^2) = x$.
3. $x^2 y + xy^2 + xy + y^2 + 3x = 0$. [*Delhi* 07, *Bundelkhand* 12]

4. $y(x-y)^3 = y(x-y) + 2$
5. $y^3 - xy^2 - x^2 y + x^3 + x^2 - y^2 - 1 = 0$. [*Raipur* 2010]
6. $(a+x)^2(b^2 + x^2) = x^2 y^2$. [*Kumaon* 2009]
7. $(x+y)^2(x+2y+2) = x + 9y + 2$
[*Jabalpur* 05, *Garhwal* 05, *Meerut* 11]
8. $xy^2 = 4a^2(2a - x)$. [*Berhampur* 2009]
9. $(x^2 - a^2)y^2 = x^2(x^2 - 4a^2)$.
[*Rewa* 04, *Agra* 05, *Rohilkhand* 05, 06, *Meerut* 06]
10. $x^2(x-y)^2 + 9(x^2 - y^2) = 9xy$. [*Kumaon* 2012]
11. $(x-y)^2(x^2 + y^2) - 10(x-y)x^2 + 12y^2 + 2x + y = 0$.
[*Kumaon* 2013]
12. $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$. [*Avadh* 2004]
13. $x^3 + 2x^2 y + xy^2 - x^2 - xy + 2 = 0$.
[*Garhwal* 09, *Kashi Vidyapeeth* 12]
14. $x^3 - 2x^2 y + xy^2 + x^2 - xy + 2 = 0$. [*Garhwal* 10, *Raipur* 15]
15. $(y-x)(y-2x)^2 + (y+3x)(y-2x) + 2x + 2y - 1 = 0$.
[*Kanpur* 10, *Bundelkhand* 14]
16. $(x^2 - 3x + 2)(x + y - 2) + 1 = 0$.
17. $x^2(x+y)(x-y)^2 + ax^3(x-y) - a^2 y^3 = 0$. [*Kumaon* 2011]
18. $(y-a)^2(x^2 - a^2) = x^4 + a^4$.
19. $y^2(x - 2a) = x^3 - a^3$.
[*Bundelkhand* 08, *Rohilkhand* 08, *Avadh* 11]
20. Find the asymptotes of the curve whose equation is

$$(x-3)(x-2)y^2 - 9x^2 = 0.$$

and determine on which side of each asymptote the corresponding branch of the curve lies.

ANSWERS

1. $y = x + \frac{1}{3}a$.
2. $x = \pm a, y = 0$
3. $x + 1 = 0, y = 0, x + y = 0$.
4. $y = x, y = 0, y = x \pm 1$
5. $y = x, y = x + 1, y = -x$.
6. $\pm y = x + a, x = 0$.
7. $x + 2y + 2 = 0, x + y = \pm 2\sqrt{2}$.
8. $x = 0$.
9. $x = \pm a, y = \pm x$.
10. $x - y = \pm 3, x = \pm 3$.
11. $x - y = 2, x - y = 3$.
12. $y = \pm x, x + 2y + 1 = 0$.

13. $x=0, x+y=0, x+y-1=0$.

14. $y=x-1$, $y=2x$, $y+x+2=0$.

15. $y=2x-2$, $y=2x-3$, $y=x+4$.

16. $x=2, x=1$, $y+x=2$.

17. $x=\pm a$, $y=x+a$, $y=\pm x-\frac{1}{2}a$.

18. $x=\pm a$, $y=x+a$, $y=-x+a$.

19. $x=2a$, $y=x+a$, $y=-x+a$.

20. $x=3, x=2$, $y=3$, $y=-3$.

Above $y=3$ in the first quadrant, below $y=3$ in the 2nd quad., above $y=-3$ in the 3rd quad., and below $y=-3$ in the 4th quad. To the left of $x=2$ and to the right of $x=3$.

9.7. Intersections of the Curve and its Asymptotes. A straight line cuts a curve of degree *n*, in general, in *n* points. As one of these points of intersection is kept fixed, and another point of intersection is made to tend to it, the straight line tends to the tangent at the former point. Hence a tangent (and therefore, as a particular case), each asymptote will, in general, cut the curve in $n-2$ points. The *n* asymptotes will therefore cut the curve in $n(n-2)$ points.

Let the asymptotes of a curve of degree *n* be

$$F_n \equiv (y-m_1x-c_1)(y-m_2x-c_2)\ldots(y-m_nx-c_n)=0$$

Let now the equation to any curve having asymptotes $F_n=0$ be put in the form

$$F_n+P=0. \qquad (1)$$

Then *P cannot be of a degree higher than* $n-2$, otherwise the terms of degree $n-1$ and higher will be different in F_n and F_n+P, and so the usual method of finding asymptotes (§ 7.5 or 7.6) will give values of m or c, or both, different from $m_1, c_1; m_2, c_2; \ldots$

Now if $S_1=0$ and $S_2=0$ represent two curves, then $S_1-\lambda S_2=0$ represents some curve through the intersections of $S_1=0$ $S_2=0$.

Therefore $F_n+P-F_n=0$, i.e. $P=0$, is some curve passing through the intersections of $F_n+P=0$ and $F_n=0$. So, as P is of a degree not higher than $n-2$, we get the following proposition:

A curve of degree n − 2, or less, can be made to pass through the n (n − 2) points of intersection of a curve of degree n and its n asymptotes.

NOTE. Of course, many other curves can be made to pass through these points, but they are generally of a higher degree than $n-2$.

Ex. 1. Find the asymptotes of the curve

$$x^2y-xy^2+xy+y^2+x-y=0.$$

and show that they cut the curve again in three points which lie on the line $x+y=0$. **[*Indore* 06, *Rohilkhand* 12]**

We have $$x^2y-xy^2+xy+y^2+x-y=0. \qquad (1)$$

The asymptotes parallel to the axes are $y=0$ and $x-1=0$.

Now from (1), we get

$$xy(x-y)+xy(1+y/x)+xy(1/y-1/x)=0$$

or $$x-y+1+y/x+1/y-1/x=0$$

When $x, y\to\infty$, then $y/x\to 1$, so the third asymptote is $x-y+2=0$.

So the joint equation of the asymptotes is

$$y(x-1)(x-y+2)=0$$

or $$x^2y-xy^2+xy+y^2-2y=0 \qquad (2)$$

Subtracting (2) from the equation (1) of the curve, we get

$$x+y=0.$$

So the points of intersection of the curve and its asymptotes lie on this line and the asymptotes cut the curve in $n(n-2)$ i.e. $3(3-2)=3$ points.

Ex. 2. Find the asymptotes of the curve

$$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$$

[*Delhi* 2003]

and show that they cut the curve in eight points which lie on a circle.

The given curve is

$$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0 \qquad (1)$$

Putting $x=1$ and $y=m$ in the 4th degree terms and equating to zero, we get

$$\phi_4(m)=0 \quad \text{i.e.} \quad (1-4m^2)(1-9m^2)=0$$

or

$$m=\pm\tfrac{1}{2},\pm\tfrac{1}{3}.$$

Then $c=-\dfrac{\phi_3(m)}{\phi'_4(m)}=-\dfrac{5(m-m^2-6m^3)}{(1-4m^2)(-18m)+(-8m)(1-9m^2)}$

$$=\frac{5(1-m-6m^2)}{26-144m^2}=\tfrac{1}{2},\ 0,\ 0,\tfrac{1}{3} \text{ respectively}$$

when

$$m=\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{3},-\tfrac{1}{3}.$$

So the asymptotes of the curve are

$$x-2y+1=0, x+2y=0, x-3y=0, x+3y-1=0.$$

Hence their joint equation is

$$(x-2y+1)(x+2y)(x-3y)(x+3y-1)=0$$

or $$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3-x^2+xy+6y^2=0 \qquad (2)$$

Subtracting (2) from the equation (1) of the curve, we get

$$x^2+y^2-1=0.$$

Hence the points of intersection of the curve and its asymptotes lie on this circle and the asymptotes cut the curve in $n(n-2)$ i.e. $4(4-2)=8$ points.

EXAMPLES

1. Show that the asymptotes of the cubic

$$x^3-2y^3+xy(2x-y)+y(x-y)+1=0$$

cut the curve in three points which lie on the straight line

$$x-y+1=0. \qquad [\textit{Kanpur } 2009]$$

2. Find the equation of the straight line on which lie three points of intersection of the cubic

$$x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0$$

and its asymptotes. [*Delhi* 2007]

3. Form the equation of the quartic curve which has $x=0$, $y=0$, $y=x$, $y=-x$, for asymptotes, which passes through the point (a,b), and which cuts its asymptotes again in eight points lying upon the circle $x^2+y^2=a^2$.

4. Show that the eight points of intersection of the curve

$$xy(x^2-y^2)+x^2+y^2=a^2$$

and its asymptotes lie on a circle whose centre is at the origin.

5. Show that the eight points of intersection of the curve

$$x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$$

and its asymptotes lie on a rectangular hyperbola. [*Indore* 2005]

6. Find the equation of the cubic which has the same asymptotes as the curve $x^3-6x^2y+11xy^2-6y^3+4x+5y+7=0$ and which passes through the points $(0,0)$, $(2,0)$ and $(0,2)$.

7. Find the equation of the cubic which has the same asymptotes as the curve $x^3-6x^2y+11xy^2-6y^3+x+y+1=0$ and which touches the axis of Y at the origin and passes through the point $(3,2)$. [*Jabalpur* 06, *Agra* 07]

ANSWERS

2. $x+3y-1=0.$.
3. The the quartic curve is
$$bxy(x^2-y^2)+a(b^2-a^2)(x^2+y^2-a^2)=0.$$
6. $x^3-6x^2y+11xy^2-6y^3-4x+24y=0.$
7. $x^3-6x^2y+11xy^2-6y^3-x=0.$

9.8. Curvilinear Asymptotes. If the equation to a curve can be put in the form

$$y=a_0x^n+a_1x^{n-1}+\ldots+a_n+A/x+B/x^2+\ldots \qquad (1)$$

then $$y=a_0x^n+a_1x^{n-1}+\ldots+a_n \qquad (2)$$

is said to be asymptotic to (1).

In particular, if the equation to the curve can be put as

$$y=ax^2+bx+c+A/x+B/x^2+\ldots,\quad a\neq 0,$$

it possesses the parabolic asymptote $\quad y=ax^2+bx+c.$

It will be observed that if two curves are asymptotic in accordance with the above definition, the difference of their ordinates for a given value of x tends to zero as $x\to\infty$. (Cf. § 9.52).

Similar definitions hold for curves in which x can be expanded in negative powers of y.

Ex. Show that the curve $ax^2y=x^4+a^2x^2+a^4$ has a parabolic asymptote.

Here $y=\frac{x^2}{a}+a+\frac{a^3}{x^2}$, so the given curve has the parabolic asymptote $ay=x^2+a^2$.

9.81. Asymptotes to non-algebraic curves. The method of substituting $y=mx+c$ and equating to zero the coefficients of the two highest powers of x applies only to algebraic curves. In the case of non-algebraic curves the asymptotes can be found in simple cases by applying the definition, or by the expansion of y in negative powers of x.

Ex. Find the asymptotes of $y=\tan x$.

Here $\frac{dy}{dx}=\sec^2 x$, hence the tangent at (x, y) is

$$Y-\tan x=(X-x)\sec^2 x.$$

or

$$Y\cos^2 x-\sin x\cos x=X-x. \qquad (1)$$

Now as $x\to\frac{1}{2}\pi$ from the left, $y\to\infty$, and the distance of (x, y) from the origin tends to infinity. Hence, to obtain the asymptote we must take the limit of (1) as $x\to\frac{1}{2}\pi$. This gives

$$Y.0-0=X-\frac{1}{2}\pi, \quad \text{i.e.} \quad X=\frac{1}{2}\pi.$$

This is one asymptote. The other asymptotes are

$$X=-\frac{1}{2}\pi,\ \pm\frac{1}{2}\pi,\ldots$$

EXAMPLES

1. Show that the curve represented by $x^3+aby-axy=0$ has a parabolic asymptote $x^2+bx+b^2=ay$.

2. Find the rectilinear and parabolic asymptotes of the curve

$$x=(y^3-a^3)/ay.$$

Find the asymptotes of

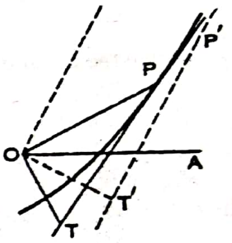
3. $y=e^{-x^2}$.
4. $y=e^{ax}$.
5. $y=\sec x$.
6. $y=\operatorname{cosec} x$.
7. $y=\log x$.

ANSWERS

2. $y=0,\ y^2=ax$.
3. $y=0$.
4. $y=0$.
5. $x=\pm\frac{1}{2}\pi,\pm\frac{3}{2}\pi,\ldots\ldots$
6. $x=0,\pm\pi,\pm 2\pi,\ldots\ldots$
7. $x=0$.

9.9. Polar Curves. *If* α *be a root of the equation* $f(\theta)=0$, *then* $r\sin(\theta-\alpha)=1/f'(\alpha)$ *is an asymptote of the curve*

$$1/r=f(\theta).$$



Let P be any point (r, θ) on

$$1/r=f(\theta). \qquad (1)$$

Let OT be the perpendicular to the radius vector OP, cutting the tangent at P in T. Then OT is the polar subtangent and so

$$OT=r^2\frac{d\theta}{dr}.$$

Now let $\theta\to\alpha$. Then $f(\theta)$ will tend to 0, because $f(\alpha)=0$ by hypothesis, and we suppose $f(\theta)$ to be a continuous function.

Therefore r will $\to\infty$ by equation (1), values of θ on one side of α alone being considered.

Thus PT will tend to the asymptote, and OT will tend to the value of $r^2 d\theta/dr$ at $\theta=\alpha$.

Also OP and PT will tend to become parallel, and thus the angle OTP will tend to a right angle (as shown by the dotted lines in the figure).

We see, therefore, that the asymptote is the straight line which is parallel to the radius vector $\theta=\alpha$, and is situated at a distance from O equal to $\left(r^2\frac{d\theta}{dr}\right)_{\theta=\alpha}$

to the right of an observer looking from the origin in the direction $\theta = \alpha$.

So the perpendicular on the asymptote, viz. *OT*, makes an angle of $\frac{1}{2}\pi$ with the radius vector $\theta = \alpha$, and therefore an angle of $\alpha - \frac{1}{2}\pi$ with the initial line.

But the polar equation of the straight line the perpendicular on which from the origin makes an angle β with the initial line and is of length p is

$$r\cos(\theta - \beta) = p.$$

Hence the equation of the asymptote must be

$$r\cos\left(\theta - \alpha + \frac{1}{2}\pi\right) = \left(r^2 \frac{d\theta}{dr}\right)_{\theta=\alpha}.$$

Substituting the value of $r^2 d\theta / dr$ from the equation of the curve, the asymptote becomes

$$\boldsymbol{r \sin(\theta - \alpha) = 1 / f'(\alpha)}$$

Ex. 1. Find the asymptotes of the curve $r(1 - e^{\theta}) = a$.

Here $\quad 1/r = (1 - e^{\theta})/a = f(\theta)$, say.

Now $\quad f(\theta) = 0, \text{ if } \theta = 0.$

Also $\quad f'(\theta) = -e^{\theta}/a = -1/a$ at $\theta = 0$.

Therefore, the asymptote is $r\sin\theta = -a$.

Ex. 2. Find the asymptotes of the curve

$$r\sin n\theta = a.$$

[*Jabalpur* 2007]

The equation to the curve can be written as

$$1/r = (1/a)\sin n\theta = f(\theta), \text{ say.}$$

Now $f(\theta) = 0$ if $\sin n\theta = 0$ i.e. $n\theta = m\pi$, where m is an integer.

Therefore $\quad \theta = m\pi / n = \alpha$, say, are the roots of $f(\theta) = 0$.

Also $\quad f'(\theta) = (1/a).n.\cos n\theta$

Therefore $\quad f'(a) = (1/a).n.\cos m\pi = (-1)^m (n/a)$.

The asymptotes, therefore, are $\quad r\sin\left(\theta - \frac{m\pi}{n}\right) = (-1)^n (a/n)$.

Ex. 3. Find the asymptotes of the curve $r = a\tan\theta$.

[*Magadh* 04, *Lucknow* 05]

The equation to the curve can be written as

$$1/r = (1/a)\cot\theta = (1/a)(\cos\theta/\sin\theta) = f(\theta), \text{ say.}$$

Then $\quad f(\theta) = 0 \quad$ gives $\quad \cos\theta = 0$.

Therefore $\quad \theta = (2n+1)\frac{1}{2}\pi = \alpha$, say, are the roots of $f(\theta) = 0$ where n is an any integer.

Also $\quad f'(\theta) = (-1/a)\operatorname{cosec}^2\theta$.

Therefore $\quad f'(\alpha) = f'[(2n+1)\pi/2] = \frac{-1}{a} \cdot \frac{1}{[\sin(2n+1)\pi/2]^2}$

$$= \frac{-1}{a(-1)^{2n}} = \frac{-1}{a}. \qquad [\text{since } (-1)^{2n} = 1 \text{ for all } n]$$

Hence the asymptotes are $r\sin\left[\theta - (2n+1)\frac{1}{2}\pi\right] = -a$

or $\quad -r\sin\left[(n\pi + \frac{\pi}{2}) - \theta\right] = -a \qquad [\text{since } \sin(-\theta) = -\sin\theta]$

or $\quad (-1)^n r\sin\left(\frac{\pi}{2} - \theta\right) = a \quad$ or $\quad r\cos\theta = a(-1)^n$.

Putting $n = 0, 1, 2, \ldots$, the asymptotes of the curve are

$$r\cos\theta = \pm a.$$

9.91. Circular asymptotes. If the equation to a curve is

$$r = f(\theta) \qquad \text{and if} \qquad \lim_{\theta\to\infty} f(\theta) = a,$$

then the circle $r = a$ is called a circular asymptote of the curve $r = f(\theta)$. It is evident that the curve approaches the circle as $\theta \to \infty$.

Ex. Find the circular asymptote of the curve $\quad r = \dfrac{\theta^3 - 1}{\theta^3 + 1}$.

[*Kashi Vidyapeeth* 2010]

Dividing by θ^3, and taking limits, we see that the circular asymptote is given by

$$r = \lim_{\theta\to\infty} \frac{1 - (1/\theta^3)}{1 + (1/\theta^3)} = 1 \qquad \text{i.e.} \qquad r = 1.$$

EXAMPLES

Find the asymptotes of the following polar curves :

1. $r=2/(1+2\sin\theta)$ 2. $r\theta=a$. [*Meerut* 2008]
3. $r=4(\sec\theta+\tan\theta)$. 4. $r\cos\theta=4\sin^2\theta$. [*Meerut* 2005]
5. $r=2\theta/\sin\theta$. 6. $r\sin\theta=a\cos 2\theta$.
7. $r\log\theta=a$. 8. $r\cos 2\theta=a\sin 3\theta$. [*Kumaon* 2003]
9. $r\sin 2\theta=a$. 10. $r=a\operatorname{cosec}\theta+b$.
11. $r^n\sin n\theta=a^n$. [*Lucknow* 2004]
12. $r\theta\cos\theta=a\cos 2\theta$.

Find the circular asymptotes of the following curves :

13. $r(\theta^2+1)=\theta^2-1$. 14. $r=(3\theta^2+2\theta+1)/(\theta^2+\theta+1)$.
15. $r(e^\theta-1)=a(e^\theta+1)$.

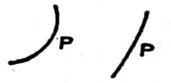
ANSWERS

1. $r\sin(\theta\pm\frac{1}{6}\pi)=2/\sqrt{3}$. **2.** $r\sin\theta=a$.
3. $r\cos\theta=8$. **4.** $r\cos\theta=4$.
5. $r\sin\theta=2k\pi;\ k=\pm1,\pm2,\ldots$ **6.** $r\sin\theta=a$.
7. $a=r\sin(\theta-1)$. **8.** $r\sin(\theta\pm\pi/4)=-a/2\sqrt{2}$.
9. $r\sin\theta=\pm\frac{1}{2}a,\ r\cos\theta=\pm\frac{1}{2}a$. **10.** $r\sin\theta=a$.
11. $n\theta=m\pi$, where m is an integer, if $n>1$:
no asymptote if $n<1$.
12. $r\sin\theta=a,\ r\cos\theta=a/(m+\frac{1}{2})\pi$.
13. $r=1$. **14.** $r=3$. **15.** $r=a$.

10

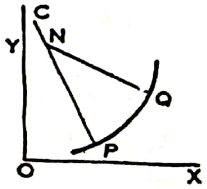
Curvature

10.1. Curvature. Of the two curves shown here, one bends more sharply than the other. In other words, one has a greater *curvature* than the other.



If P be a given point on the curve, and the curve in a small neighbourhood of P be regarded roughly as an arc of a circle, we notice from the figure that the *radius of such a circle would be small when the curvature is great, and vice versa.* Thus, we see that *curvature varies inversely as the radius.*

10.11. Definitions. Let P be a given point on a curve and Q any other neighbouring point on it. Let the normals at P and Q intersect in N. If N tends to a definite position C as $Q\to P$, then C is called the ***centre of curvature*** of the curve at P.



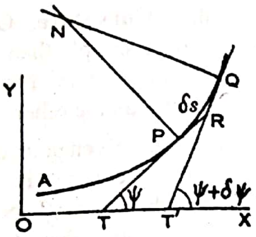
N must tend to C whether $Q\to P$ from the right or from the left.

The reciprocal of the distance CP is called the ***curvature*** of the curve at P. The circle with its centre at C and radius CP is called the ***circle of curvature*** of the curve at P.

The distance CP is called the ***radius of curvature*** of the curve at P. The radius of curvature is usually denoted by the Greek letter ρ. Any chord, drawn through P, of the circle of curvature at P, is called a ***chord of curvature.***

~~10~~.12. A formula for radius of curvature. Let P be a given point on the curve APQ, *and let* Q be any other point on the curve near it.

Let the length of the arc up to P, measured from a fixed point A on the curve, be s and let the length from A up to Q be $s + \delta s$ i.e. arc $PQ = \delta s$.



Let the tangent PT at P to the curve make with any line (which might be taken to be the axis of x), the $\angle \psi$, and let the tangent QT' at Q make with the same line the $\angle \psi + \delta\psi$. Let these tangents meet in R.

Then, the angle between the two tangents $\angle TRT' = \delta\psi$.

Let the normals at P and Q be PN and QN respectively, N being their point of intersection. Join PQ.

By geometry, the angle between two lines is the same as the angle between their tangents, so $\angle PNQ = \delta\psi$. (1)

Now from the triangle PNQ,

$$\frac{PN}{\text{chord } PQ} = \frac{\sin NQP}{\sin PNQ} = \frac{\sin NQP}{\sin \delta\psi} \quad [\text{using (1)}] \qquad (2)$$

Also as $Q \to P$, $\delta s \to 0$, $\delta\psi \to 0$. Then the radius of curvature at P is given by

$$\rho = \lim_{\delta s \to 0} PN = \lim_{\delta\psi \to 0} \frac{\text{chord } PQ \sin NQP}{\sin \delta\psi} \quad [\text{from (2)}]$$

$$= \lim_{\delta\psi \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \cdot \sin NQP. \qquad (3)$$

But as $\delta\psi \to 0$, the point $Q \to P$ and arc $PQ \to$ chord PQ so that chord $PQ / \delta s \to 1$ and $\delta\psi / \sin \delta\psi \to 1$, as is well known.

Also, $\angle NQP \to \frac{1}{2}\pi$, because $\angle NQP = \frac{1}{2}\pi - \angle PQT'$ and $\angle PQT' \to 0$ when $\delta\psi \to 0$.

Thus, (3) gives $\qquad \rho = \lim_{\delta\psi \to 0} \dfrac{\delta s}{\delta\psi}$.

Hence $\qquad \rho = \dfrac{ds}{d\psi}$.

The relation between s and ψ for any curve is called the ***intrinsic equation*** of the curve. If we consider any arc PQ, the angle between the tangents to the curve at P and Q is called the ***angle of contingence*** of the arc PQ.

The above proof shows that we might say that at P

$$\rho = \lim_{PQ \to 0} \left(\frac{\text{arc } PQ}{\text{the angle of contingence of the arc PQ}} \right).$$

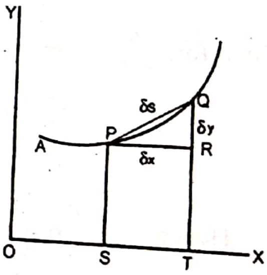
Ex. Find ρ for the catenary whose intrinsic equation is

$$s = c \tan\psi.$$

$\rho \frac{ds}{d\psi} = c\sec^2\psi$

We have $\qquad \rho = ds / d\psi = c \sec^2 \psi$.

10.2. Derivative of the length of arc. Let $P(x, y)$ be a given point on the curve APQ, and let $Q(x + \delta x, y + \delta y)$ be any other point on the curve near it.



As in the previous article, let the length of the arc up to P, measured from a fixed point A on the curve, be s and let the length from A up to Q be $s + \delta s$.

Then, arc $PQ = \delta s$ (1)

From the right-angled triangle PQR, we have

$$PQ^2 = PR^2 + RQ^2 = (\delta x)^2 + (\delta y)^2$$

Hence $\qquad \left(\dfrac{PQ}{\delta x}\right)^2 = 1 + \left(\dfrac{\delta y}{\delta x}\right)^2$

i.e. $\qquad \left(\dfrac{\text{chord } PQ}{\delta x}\right)^2 = 1 + \left(\dfrac{\delta y}{\delta x}\right)^2 \qquad (2)$

Now as $\delta x \to 0$, the point $Q \to P$ and arc $PQ \to$ chord PQ. Therefore from (1),

$$\text{arc } PQ = \text{chord } PQ = \delta s \qquad (3)$$

From (2) and (3), we have $\left(\dfrac{\delta s}{\delta x}\right)^2 = 1 + \left(\dfrac{\delta y}{\delta x}\right)^2$

Taking the limit of both sides as $\delta x \to 0$, we get

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

or

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

taking the positive sign before the radical since by convention, s is measured positively in the direction of x increasing i.e. s increases with x so that ds/dx is positive.

~~10.3.~~ Cartesian Formula for Radius of Curvature.

We know that $\dfrac{dy}{dx} = \tan\psi$.

Therefore $\dfrac{d^2y}{dx^2} = \sec^2\psi \cdot \dfrac{d\psi}{dx} = \sec^2\psi \cdot \dfrac{d\psi}{ds} \cdot \dfrac{ds}{dx}$

$$= \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \cdot \frac{1}{\rho} \cdot \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2}.$$

Hence

$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2}.$$

We have not considered the sign of ρ. It is customary to attach that sign to the radical which would give a positive sign to ρ.

The definition of the radius of curvature shows that its value depends onlyon the curve and not on the axes. Hence, interchanging the axes of x and y, we obtain

$$\rho = \frac{\{1 + (dx/dy)^2\}^{3/2}}{d^2x/dy^2}.$$

which is useful when the tangent is parallel to the y-axis.

Ex. 1. Show that for the curve $s^2 = 8ay$, $\rho = 4a\sqrt{1 - (y/2a)}$.

[***Kanpur*** **09,** ***Lucknow*** **12**]

Differentiating the given curve w.r.t. s, we have

$$2s = 8a \cdot dy/ds = 8a\sin\psi \text{ since } dy/ds = \sin\psi \quad [\S\ 8.111\ (2)]$$

or $\quad s = 4a\sin\psi$ which differentiated w.r.t. ψ is $ds/d\psi = 4a\cos\psi$.

Hence $\quad \rho = ds/d\psi = 4a\cos\psi = 4a\sqrt{1 - \sin^2\psi}$

$$= 4a\sqrt{1 - (s^2/16a^2)} = 4a\sqrt{1 - (8ay/16a^2)}$$

$$= 4a\sqrt{1 - (y/2a)}.$$

$\rho = \{1 + (dy/dx)^2\}^{3/2} / d^2y/dx^2$

Ex. 2. Show that the curvature at the point $(3a/2, 3a/2)$ on the Folium $x^3 + y^3 = 3axy$ is $-3a/8\sqrt{2}$.

[***Rajasthan*** **03,** ***Delhi*** **03,** ***Garhwal*** **06**]

Differentiating w.r.t. x gives $\quad 3x^2 + 3y^2y' = 3ay + 3axy'$. (1)

where dashes denote differentiation with respect to x.

Therefore $\quad y' = (ay - x^2)/(y^2 - ax) = -1$ at $(3a/2, 3a/2)$.

Dividing (1) by 3 and differentiating again, we have

$$2x + 2yy'^2 + y^2y'' = ay' + ay' + axy''.$$

At the point $(3a/2, 3a/2)$, this gives

$$3a + 3a(-1)^2 + (3a/2)^2\,y'' = 2a(-1) + (3a^2/2)\,y'',$$

giving

$$y'' = -32/3a.$$

Therefore, at the point $(3a/2, 3a/2)$

$$\rho = (1 + y'^2)^{3/2}/y'' = (1+1)^{3/2}/(-32/3a) = -3a\sqrt{2}/16$$

or

$$\rho = -3a/8\sqrt{2}.$$

Ex. 3. Prove that for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\rho = \dfrac{a^2b^2}{p^3}$, p being the perpendicular from the centre upon the tangent at (x, y).

[***Indore*** **04,** ***Kumaon*** **05,** ***Avadh*** **05,** ***Meerut*** **07,** ***Jabalpur*** **09,** ***Raipur*** **14**]

The given ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $b^2x^2 + a^2y^2 = a^2b^2 \qquad (1)$

Differentiating (1) with respect to x, we have

$$(2x)b^2 + a^2(2y)\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

Differentiating this again, we get

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\left\{\frac{y-x\,(dy/dx)}{y^2}\right\}=-\frac{b^2}{a^2y^2}\left\{y+x\left(\frac{b^2x}{a^2y}\right)\right\}$$

$$=-\frac{b^2}{a^2y^2}\left(\frac{a^2y^2+b^2x^2}{a^2y}\right)=-\frac{b^4}{a^2y^3}. \qquad \text{[using (1)]}$$

Hence $\rho=\dfrac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2}=\dfrac{(1+b^4x^2/a^4y^2)^{3/2}}{b^4/a^2y^3}$

ignoring the minus sign

$$=\frac{(a^4y^2+b^4x^2)^{3/2}}{(b^4/a^2y^3)\,a^6y^3}=\frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}. \qquad (2)$$

The equation of the tangent to the ellipse at (x, y) is given by

$$\frac{xX}{a^2}+\frac{yY}{b^2}=1. \qquad (3)$$

Then the length of the perpendicular on (3) from the centre of the ellipse (0, 0) is

$$p=\frac{1}{\sqrt{(x^2/a^4+y^2/b^4)}}=\frac{a^2b^2}{\sqrt{(b^4x^2+a^4y^2)}}$$

or $\sqrt{(b^4x^2+a^4y^2)}=(a^2b^2/p).$

So $(b^4x^2+a^4y^2)^{3/2}=(a^2b^2/p)^3=a^6b^6/p^3. \qquad (4)$

Using (4) in (2), we have, $\rho=\dfrac{a^6b^6/p^3}{a^4b^4}=\dfrac{a^2b^2}{p^3}.$

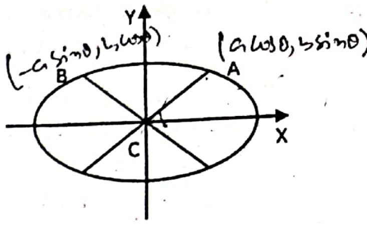
Ex. 4. If ρ_1, ρ_2 be the radii of curvature at the extremities of two conjugate diameters of an ellipse, prove that

$$(\rho_1^{2/3}+\rho_2^{2/3})\,a^{2/3}b^{2/3}=a^2+b^2.$$

[*Rewa* 05, *Bundelkhand* 06, *Bilaspur* 07, *Lucknow* 07, *Jabalpur* 08, *Kashi Vidyapeeth* 10, *Meerut* 04, 06, 11, *Kanpur* 11]

As in Ex. 3, the radius of curvature at any point (x, y) of the ellipse $x^2/a^2+y^2/b^2=1$ is given by

$$\rho=\frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}. \qquad (1)$$



Let CA and CB be the two conjugate diameters of the given ellipse with centre C (0, 0). Then from Coordinate Geometry, the coordinates of the extremities of the two conjugate diameters will be given by $A\,(a\cos\theta, b\sin\theta)$ and $B\,(-a\sin\theta, b\cos\theta)$, θ being the eccentric angle of the point A and $\theta+\frac{1}{2}\pi$ of B.

So, from (1), the radius of curvature at $A\,(a\cos\theta, b\sin\theta)$ is

$$\rho_1=\frac{(a^4b^2\sin^2\theta+b^4a^2\cos^2\theta)^{3/2}}{a^4b^4}=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab} \qquad (2)$$

Again from (1), the radius of curvature at $B\,(-a\sin\theta, b\cos\theta)$ is

$$\rho_2=\frac{(a^4b^2\cos^2\theta+b^4a^2\sin^2\theta)^{3/2}}{a^4b^4}=\frac{(a^2\cos^2\theta+b^2\sin^2\theta)^{3/2}}{ab} \qquad (3)$$

From (2) and (3), we have

$$\rho_1^{2/3}+\rho_2^{2/3}=\frac{a^2\sin^2\theta+b^2\cos^2\theta}{(ab)^{2/3}}+\frac{a^2\cos^2\theta+b^2\sin^2\theta}{(ab)^{2/3}}$$

$$=\frac{a^2(\sin^2\theta+\cos^2\theta)+b^2(\cos^2\theta+\sin^2\theta)}{a^{2/3}b^{2/3}}=\frac{a^2+b^2}{a^{2/3}b^{2/3}}.$$

or $(\rho_1^{2/3}+\rho_2^{2/3})\,a^{2/3}b^{2/3}=a^2+b^2.$

EXAMPLES

Find the radius of curvature at the point (s, ψ) on the following curves:

1. $s=8a\sin^2\frac{1}{6}\psi$ (Cardioid).
2. $s=c\log\sec\psi$ (Tractrix). **[*Agra* 04, *Kashi Vidyapeeth* 12]**
3. $s=4a\sin\psi$ (Cycloid).

[*Rohilkhand* 08, *Kashi Vidyapeeth* 11]

4. Prove that for the curve
   (i) $s=a\log\tan(\frac{1}{4}\pi+\frac{1}{2}\psi)+a\tan\psi\sec\psi,\ \rho=2a\sec^3\psi.$ **[*Avadh* 08, *Lucknow* 08]**
   (ii) $s=a\log\cot(\frac{1}{4}\pi-\frac{1}{2}\psi)+a\sin\psi\sec^2\psi,\ \rho=2a\sec^3\psi.$ **[*Kanpur* 2012]**

Find the radius of curvature of the following curves:

5. $y=e^x$, at the point where it crosses the y-axis. [*Magadh* 2002]

6. $\sqrt{x}+\sqrt{y}=1$ at $(\frac{1}{4},\frac{1}{4})$. [*Agra* 2010]

Find the radius of curvature at the point (x, y) on the following curves

7. $y^2=4ax$. 8. $ay^2=x^3$. 10. $xy=k^2$.

10. $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$. [*Agra* 2007]

11. $x^{2/3}+y^{2/3}=1$. [*Patna* 06, *Kanpur* 12, *Kashi Vidyapeeth* 12]

12. $y=c\log\sec(x/c)$

[*Rewa* 04, *Kanpur* 07, *Purvanchal* 09, *Kashi Vidyapeeth* 09]

13. Prove that at the point $x=\frac{1}{2}\pi$ of the curve

$$y=4\sin x-\sin 2x,\quad \rho=5\sqrt{5}/4.$$

14. In the ellipse $x^2/a^2+y^2/b^2=1$, show that the radius of curvature at an end of the major axis is equal to the semi - latus rectum of the ellipse.

15. If CP, CD be a pair of conjugate semi-diameters of an ellipse; prove that the radius of curvature at P is CD^3/ab, a and b being the lengths of the semi-axes of the ellipse. [*Rohilkhand* 2011]

16. If ρ_1 and ρ_2 be the radii of curvature at the extremities of a focal chord of a parabola $y^2=4ax$, prove that

$$(\rho_1)^{-2/3}+(\rho_2)^{-2/3}=(2a)^{-2/3}.$$ [*Kashi Vidyapeeth* 2008]

17. For the curve $y=\dfrac{ax}{a+x}$, if ρ is the radius of curvature at any point (x, y), show that $(2\rho/a)^{2/3}=(y/x)^2+(x/y)^2$.

[*Kumaon* 08, 09]

ANSWERS

1. $\frac{4}{3}a\sin\frac{1}{3}\psi$. **2.** $c\tan\psi$.
3. $4a\cos\psi$. **5.** $\sqrt{8}$.
6. $1/\sqrt{2}$. **7.** $(2/\sqrt{a})(x+a)^{3/2}$.
8. $(1/6a)(4a+9x)^{3/2}x^{1/2}$. **9.** $(x^2+y^2)^{3/2}/2k^2$.
10. y^2/a. **11.** $3(axy)^{1/3}$. **12.** $c\sec(x/c)$.

10.31. Radius of Curvature. Parametric Equations. et a curve be defined by the parametric equations $x=f(t)$ and $y=\phi(t)$ Let dashes denote differentiations with respect to t, then

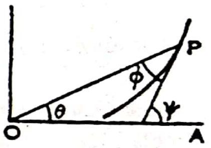
$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{y'}{x'}$$

and
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y'}{x'}\right)=\frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx}=\frac{y''x'-y'x''}{x'^2}\cdot\frac{1}{x'}.$$

Hence, we have

$$\rho=\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}=\frac{\left\{1+\left(\dfrac{y'}{x'}\right)^2\right\}^{3/2}}{\dfrac{y''x'-y'x''}{x'^3}}=\frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}.$$

10.32. Radius of Curvature. Pedal Equations. Let the pedal equation of the curve be $r=f(p)$. The relation between θ, ϕ and ψ (as is evident from the figure) is



$$\psi=\theta+\phi. \qquad (1)$$

Differentiating (1) w.r.t. s, we get

$$\frac{d\psi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{dr}\cdot\frac{dr}{ds}$$

or
$$\frac{1}{\rho}=\frac{1}{r}\sin\phi+\cos\phi\frac{d\phi}{dr}$$

since
$$\sin\phi=r\frac{d\theta}{ds},\quad \cos\phi=\frac{dr}{ds}.$$

or
$$\frac{1}{\rho}=\frac{1}{r}\left(\sin\phi+r\cos\phi\frac{d\phi}{dr}\right)=\frac{1}{r}\cdot\frac{d}{dr}(r\sin\phi)=\frac{1}{r}\cdot\frac{dp}{dr}.$$

Hence
$$\rho=r\frac{dr}{dp}.$$

10.33. Radius of Curvature. Polar Equations. Let the polar equation of the curve be $r = f(\theta)$. We know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

By differentiation w.r.t. r, we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\cdot 2\cdot\frac{dr}{d\theta}\cdot\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)\cdot\frac{d\theta}{dr}$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\frac{d^2r}{d\theta^2}.$$

or
$$\frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^5}\left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}$$

Hence
$$\rho = r\frac{dr}{dp} = \frac{r(1/p^3)}{\frac{1}{r^5}\{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\}}$$

Therefore
$$\rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}.$$

10.34. Alternative Formula. By putting $1/u$ for r in the formula for polar equations, and remembering that $dr/d\theta = -u^{-2}du/d\theta$, we can show easily that

$$\rho = \frac{(u^2 + u'^2)^{3/2}}{u^3(u + u'')}.$$

10.35. Radius of Curvature. Tangential Polar Form. We can also derive a formula for use when the relation between p and ψ is given. Let the tangential polar equation of the curve be $p = f(\psi)$. We have

$$\frac{dp}{d\psi} = \frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi} = \frac{dp}{dr}\cdot\cos\phi\cdot\rho = \frac{dp}{dr}\cdot\cos\phi\cdot r\frac{dr}{dp}$$

$$= r\cos\phi.$$

(by §§ 10.12., 10.32 and since $dr/ds = \cos\phi$)

We know that $p = r\sin\phi$, therefore

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi = r^2.$$

Differentiating with respect to p,

$$2p + 2\frac{dp}{d\psi}\cdot\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp} = 2r\frac{dr}{dp} = 2\rho \quad \text{(by § 10.32)}$$

or
$$\rho = p + \frac{d^2p}{d\psi^2}.$$

Ex. 1. For the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$, prove that the radius of curvature $\rho = 4a\cos\frac{1}{2}t$.

[*Utkal* 03, *Kumaon* 02, 10, *Garhwal* 07, 10, *Lucknow* 09, 10, *Agra* 11, *Bundelkhand* 07, 12, 14]

Here $\dfrac{dx}{dt} = a(1 + \cos t)$, $\dfrac{dy}{dt} = a\sin t$,

Hence
$$\frac{dy}{dx} = \frac{\sin t}{(1 + \cos t)} = \frac{2\sin\frac{1}{2}t\cos\frac{1}{2}t}{2\cos^2\frac{1}{2}t} = \tan\tfrac{1}{2}t.$$

Therefore
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{\frac{d}{dt}(\tan\frac{1}{2}t)}{a(1 + \cos t)} = \frac{\sec^2\frac{1}{2}t\cdot(\frac{1}{2})}{a\cdot 2\cos^2\frac{1}{2}t} = \frac{1}{4\cdot a\cos^4\frac{1}{2}t}$$

Hence
$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{\{1 + \tan^2\frac{1}{2}t\}^{3/2}}{1/(4a\cos^4\frac{1}{2}t)}$$

$$= \sec^3\tfrac{1}{2}t\cdot 4a\cos^4\tfrac{1}{2}t = 4a\cos\tfrac{1}{2}t.$$

Ex. 2. If the pedal equation of a curve is $p^2 = ar$, find ρ.

By differentiation, $2p = a\,dr/dp$.

Therefore $\rho = r\dfrac{dr}{dp} = 2rp/a = 2p^3/a^2$.

Ex. 3. Show that for the curve $r^n = a^n \cos n\theta$, the radius of curvature is $\dfrac{a^n}{(n+1)r^{n-1}}$. [*Rewa* 03, *Patna* 03, *Indore* 07, *Magadh* 10]

By logarithmic differentiation, we have

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{n\sin n\theta}{\cos n\theta} \quad \text{i.e.} \quad \frac{dr}{d\theta} = -r\tan n\theta.$$

Therefore
$$\frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta}\tan n\theta - nr\sec^2 n\theta$$
$$= r\tan^2 n\theta - nr\sec^2 n\theta.$$

Hence
$$\rho = \frac{\{r^2 + r^2\tan^2 n\theta\}^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$
$$= \frac{r^3\sec^3 n\theta}{(n+1)r^2\sec^2 n\theta} = \frac{r}{(n+1)\cos n\theta} = \frac{a^n}{(n+1)r^{n-1}}.$$

Ex. 4. Prove that for the cardioid $r = a(1+\cos\theta)$, ρ^2/r is constant. [*Avadh* 03, *Garhwal* 04]

Here
$$r = a(1+\cos\theta) \tag{1}$$
$$\therefore \quad dr/d\theta = -a\sin\theta \text{ and } d^2r/d\theta^2 = -a\cos\theta \tag{2}$$

$$\therefore \quad \rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}$$
$$= \frac{\{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2\}^{3/2}}{a^2(1+\cos\theta)^2 + 2(-a\sin\theta)^2 - a(1+\cos\theta)(-a\cos\theta)}$$

[using (1) & (2)]

$$= \frac{a\{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}^{3/2}}{(1+2\cos\theta+\cos^2\theta) + 2\sin^2\theta + (\cos\theta + \cos^2\theta)}$$
$$= \frac{a(2+2\cos\theta)^{3/2}}{1 + 3\cos\theta + 2\cos^2\theta + 2\sin^2\theta} = \frac{2^{3/2}.a(2\cos^2\frac{1}{2}\theta)^{3/2}}{3(2\cos^2\frac{1}{2}\theta)}$$
$$= \tfrac{4}{3}a\cos\tfrac{1}{2}\theta.$$

Now
$$\rho^2/r = \tfrac{16}{9}a^2\cos^2\tfrac{1}{2}\theta / a(1+\cos\theta)$$
$$= \tfrac{16}{9}a\cos^2\tfrac{1}{2}\theta / 2\cos^2\tfrac{1}{2}\theta$$

or $\rho^2/r = \frac{8}{9}a$ which is constant.

EXAMPLES

Find the radius of curvature at the point (p, r) on the following curves :

1. $r^3 = 2ap^2$ (Cardioid) 2. $pr = a^2$ (Hyperbola)
3. $r^3 = a^2p$ (Lemniscate) 4. $pa^n = r^{n+1}$ (Sine Spiral)
5. $p^2 = r^4/(r^2+a^2)$ (Archimedian Spiral)
6. $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$ (Ellipse)

7. Find the curvature at any point θ on the curve $x = a(\cos\theta + \log\tan\frac{1}{2}\theta)$, $y = a\sin\theta$. [*Bundelkhand* 2009]

8. Show that the radius of curvature at a point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$.

[*Meerut* 05, *Indore* 06, *Raipur* 13, *Bundelkhand* 13]

9. Find the radius of curvature for the curve $r = a(1-\cos\theta)$.

[*Gorakhpur* 03, *Bilaspur* 08, *Avadh* 09, 10, *Raipur* 12]

Find the radius of curvature at the point (r, θ) of each of the following curves:

10. $r = 2a\cos\theta$. [*Kanpur* 2006]
11. $r^2\cos 2\theta = a^2$. [*Agra* 10, *Bundelkhand* 11]
12. $r(1+\cos\theta) = a$. 13. $r = a(2\cos\theta - 1)$.
14. $r^n = a^n\sin n\theta$. [*Agra* 06, *Rohilkhand* 12]
15. $\theta = a^{-1}(r^2-a^2)^{1/2} - \cos^{-1}(a/r)$ [*Kanpur* 2011]
16. Find the radius of curvature at any point (r, θ) of the curve $r = ae^{\theta\cot\alpha}$, [*Avadh* 2004] and show that it subtends a right angle at the pole.

16. Show that at the points in which the Archimedean spiral $r = a\theta$ intersects the hyperbolical spiral $r\theta = a$, their curvatures are in the ratio 3:1.

[*Garhwal* 05]

17. Prove that for any curve, $\dfrac{r}{\rho} = \sin\phi\left(1 + \dfrac{d\phi}{d\theta}\right)$, where ρ is the radius of curvature and $\tan\phi = r\,d\theta/dr$. [*Garhwal* 04, *Lucknow* 10]

$p = r\sin\phi$

$\rho = r\dfrac{dr}{dp}$

18. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1+\cos\theta)$, which passes through the pole, then show that $9(\rho_1^{2/3} + \rho_2^{2/3}) = 16a^2$. [*Jabalpur* 03, *Kanpur* 08]

19. Show that for the epi-cycloid $p = a \sin b\psi$, ρ varies as p.

20. Find the radius of curvature at any point of the ellipse $p^2 = a^2\cos^2\psi + b^2\sin^2\psi$. [*Bundelkhand* 2005]

ANSWERS

1. $\frac{2}{3}\sqrt{2ar}$. **2.** r^3/a^2. **3.** $a^2/3r$.
4. $a^n r^{-n+1}/(n+1)$. **5.** $(r^2+a^2)^{3/2}/(r^2+2a^2)$.
6. a^2b^2/p^3. **7.** $a \cot\theta$. **9.** $\frac{2}{3}\sqrt{2ar}$.
10. a. **11.** r^3/a^2. **12.** $\sqrt{8r^3}/\sqrt{a}$.
13. $a^{1/2}(3a-2r)^{3/2}/3(2a-r)$. **14.** $a^n r^{-n+1}/(n+1)$.
15. $\sqrt{r^2-a^2}$. **16.** $\rho = r \operatorname{cosec}\alpha$. **20.** a^2b^2/p^3.

10.36. Miscellaneous formulae for ρ.

(i) When x and y are given as functions of s, we know that $\cos\psi = \dfrac{dx}{ds}$.

Differentiating w.r.t. s, $\quad -\sin\psi \cdot \dfrac{d\psi}{ds} = \dfrac{d^2x}{ds^2}$

or
$$-\sin\psi \cdot \frac{1}{\rho} = \frac{d^2x}{ds^2} \qquad \left[\text{since } \rho = \frac{ds}{d\psi}\right] \tag{1}$$

Therefore
$$\rho = -\left(\frac{dy}{ds}\right) \Big/ \left(\frac{d^2x}{ds^2}\right). \qquad \left(\text{since } \sin\psi = \frac{dy}{ds}\right)$$

(ii) Similarly, starting with $\sin\psi = dy/ds$, differentiating w.r.t. s, we get $\cos\psi \cdot \dfrac{1}{\rho} = \dfrac{d^2y}{ds^2}$ (2)

or
$$\rho = \left(\frac{dx}{ds}\right) \Big/ \left(\frac{d^2y}{ds^2}\right). \qquad \left(\text{since } \cos\psi = \frac{dx}{ds}\right)$$

(iii) Squaring and adding the values of (1) and (2) obtained from above, we have
$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

10.4. Radius of Curvature at the origin.

If y can be easily expanded in powers of x by algebraic or trigonometric methods, and
$$y = px + qx^2/2! + \ldots,$$
so that the curve passes through the origin, we know that
$$p = \left(\frac{dy}{dx}\right)_{x=0,\, y=0}, \quad q = \left(\frac{d^2y}{dx^2}\right)_{x=0,\, y=0} \tag{1}$$

Hence the radius of curvature at the origin can be found by substituting (1) in the value of ρ obtained from the formula of §10.3. Then
$$\boldsymbol{\rho \text{ (at the origin)} = (1+p^2)^{3/2}/q.}$$

10.41. Newtonian Method.

If a curve $y = f(x)$ passes through the origin, and the axis of x is the tangent at the origin, we have the following values for x, y and dy/dx at the origin:
$$x = 0,\ y = 0,\ dy/dx = 0.$$

Thus, expanding by Maclaurin's Theorem,
$$y = 0 + 0.x + qx^2/2! + \ldots.$$

Dividing by x^2 and taking limits, we get
$$\lim_{x\to 0}(2y/x^2) = q.$$

Therefore by § 10.4, we have at the origin
$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{1}{q} = \lim_{x\to 0} \frac{x^2}{2y}. \tag{1}$$

(vi) Similarly, if a curve passes through the origin, and the axis of y is the tangent there, the value of ρ at the origin is
$$\lim_{x\to 0} \frac{y^2}{2x}. \tag{2}$$

Formulae (1) and (2) are known as **Newton's formulae.**

Ex. 1. Find the curvature at the origin of the curve
$$a_1x + a_2y + b_1x^2 + b_2xy + b_3y^2 + c_1x^3 + \ldots = 0.$$

Substituting for y its Maclaurin's expansion $px + qx^2/2! + \ldots$, we have the identity

$$a_1\, x + a_2\,(p\,x + q\,x^2/2! + \ldots) + b_1 x^2$$
$$+\, b_2\, x(p\,x + \ldots) + b_3\,(p\,x + \ldots)^2 + \ldots = 0.$$

Equating to zero the coefficient of x and x^2, we get

$$a_1 + a_2\, p = 0,$$

and
$$\tfrac{1}{2} a_2\, q + b_1 + b_2\, p + b_3\, p^2 = 0.$$

Hence
$$p = -a_1/a_2;\quad q = -2\frac{b_1 + b_2\, p + b_3\, p^2}{a_2}.$$

Therefore
$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{1}{2}\frac{(a_1^2 + a_2^2)^{3/2}}{b_1 a_2^2 - b_2 a_1 a_2 + b_3 a_1^2}.$$

Ex. 2. Find the radius of curvature at the origin for $x^4 - y^4 + x^3 - y^3 + x^2 - y^2 + y = 0.$ [*Delhi* 2005]

The curve passes through the origin and the tangent at the origin is the axis of x (by equating to zero the lowest degree term in the equation of the curve, i.e. $y = 0$). Hence, Newton's formula can be applied so that

$$\rho = \lim_{x\to 0}\frac{x^2}{2y} \quad \text{or} \quad 2\rho = \lim_{x\to 0}\frac{x^2}{y}.$$

Dividing the equation to the curve by y, we get

$$x^2(x^2/y) - y^3 + x(x^2/y) - y^2 + (x^2/y) - y + 1 = 0.$$

At the origin $x = y = 0$, so taking limits as $x \to 0$, we have,

$$0.2\rho - 0 + 0.2\rho - 0 + 2\rho - 0 + 1 = 0 \quad \text{or} \quad 2\rho + 1 = 0.$$

Hence (numerically) $\rho = 1/2$.

Ex. 3. Find the curvature at the origin of the curve

$$5x^3 + 7y^3 + 4x^2 y + x\,y^2 + 2x^2 + 3x\,y + y^2 + 4x = 0.$$

It is easy to verify that the curve passes through the origin and the tangent at the origin is the axis of y (by equating to zero the lowest degree term in the equation of the curve, i.e. $4x = 0$).

Hence, by Newton's formula,

$$\rho = \lim_{x\to 0}\frac{y^2}{2x} \quad \text{or} \quad 2\rho = \lim_{x\to 0}\frac{y^2}{x} \qquad (1)$$

Dividing the equation to the curve by x, we get

$$5x^2 + 7y(y^2/x) + 4x\,y + y^2 + 2x + 3y + (y^2/x) + 4 = 0.$$

Taking limits as $x \to 0$, we have,

$$\lim_{x\to 0}(y^2/x) + 4 = 0 \quad \text{or} \quad 2\rho + 4 = 0$$

using (1) and keeping in mind that at the origin $x = y = 0$.

Hence (numerically) $\rho = 2$.

Ex. 4. Find the radius of curvature at $(r,\ \theta)$ on the curve $r\theta^2 = a$.

Here
$$a\,u = \theta^2,\ a\,u' = 2\theta,\ a\,u'' = 2$$

Hence
$$\rho = \frac{(u^2 + u'^2)^{3/2}}{u^3(u + u'')} = a\frac{(\theta^4 + 4\theta^2)^{3/2}}{\theta^6(\theta^2 + 2)} = a\frac{(\theta^2 + 4)^{3/2}}{\theta^3(\theta^2 + 2)}.$$

Ex. 5. Apply Newton's formula to find the radius of curvature at the origin for the cycloid

$$x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta)$$

Here $\theta = 0$ gives $x = y = 0$. So the curve passes through the origin.

Now
$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta,$$

So,
$$\frac{dy}{dx} = \frac{dy}{d\theta} \quad \text{or} \quad \frac{dy}{dx} = \tan\tfrac{1}{2}\theta = 0 \text{ at the origin where } \theta = 0.$$

So the initial line is the tangent at the origin.

Hence, applying Newton's formula, at the origin

$$\rho = \lim_{x\to 0}\frac{x^2}{2y} = \lim_{\theta\to 0}\frac{a^2(\theta + \sin\theta)^2}{2a(1 - \cos\theta)} \qquad (1)$$

$$= \lim_{\theta\to 0}\frac{a(2\theta - \frac{1}{6}\theta^3 + \ldots.)^2}{2(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4 + \ldots.)}$$

$$= \lim_{\theta\to 0}\frac{a\theta^2(2 - \frac{1}{6}\theta + \ldots.)^2}{\theta^2(1 - \frac{1}{12}\theta^2 + \ldots.)} = 4a$$

Alternatively, (1) can be solved as an indeterminate form [0 / 0]. [See Chapter XVI] .

EXAMPLES

1. For the curve $x = c\log\{s + \sqrt{c^2 + s^2}\},\ y = \sqrt{c^2 + s^2}$, determine the radius of curvature. [*Jabalpur* 2003]

2. Show that for the curve in which $s = c\,e^{x/c}$,
$$c\rho = s(s^2 - c^2)^{1/2}.$$

3. Prove that for any curve $\frac{1}{\rho}=\frac{d}{dx}\frac{dy}{dx}$. [*Rohilkhand* 2004]

Find the radius of curvature at the origin of the following curves :

4. $y=x^4-4x^3-18x^2$.

5. $y=x^3+5x^2+6x$.

6. $y^2-3xy-4x^2+x^3+x^4y+y^5=0$. [*Jabalpur* 2004]

7. $y^2(a-x)=x^2(a+x)$. [*Kumaon* 2013]

8. Find the radius of curvature at the origin of the two branches of the curve given by the equations $y=t-t^5$, $x=1-t^2$.

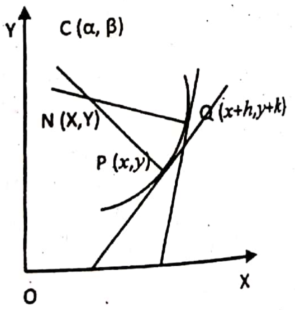
9. Show that the radius of curvature of the lemniscate $(x^2+y^2)^2=a^2(x^2-y^2)$, at the point where the tangent is parallel to the x-axis is $\sqrt{2}\,a/3$.

10. Show that the radii of curvature at the origin of the curves $x^3+y^3=3axy$ is each equal to $3a/2$. [*Kumaon* 2012]

ANSWERS

1. $(c^2+s^2)/c$.
4. 1/36.
5. $37/\sqrt{37}/10$.
6. $85\sqrt{17}/2;\ 5\sqrt{2}$
7. $\pm a\sqrt{2}$
8. $2\sqrt{2}$ for each.

10.5. Centre of Curvature. Let P be the point (x,y) and Q the point $(x+h, y+k)$. Let the normals at P and Q intersect in $N\,(X,Y)$. Let the centre of curvature for P be $C\,(\alpha,\beta)$.



The normal at P is $(Y-y)\phi(x)+X-x=0$. (1)
where $\phi(x)$ stands for *dy/dx*.

The normal at Q is

$$(Y-y-k)\phi(x+h)+X-x-h=0. \qquad (2)$$

To find the ordinate of the point of intersection of these normals, subtract (1) from (2). We have

$$(Y-y)\{\phi(x+h)-\phi(x)\}-k\,\phi(x+h)-h=0.$$

Dividing by h,

$$(Y-y)\frac{\phi(x+h)-\phi(x)}{h}-\phi(x+h).\frac{k}{h}-1=0. \qquad (3)$$

Now as $Q\to P, h\to 0,\ Y\to\beta$ by definition (see §10.11)

So $\frac{\phi(x+h)-\phi(x)}{h}\to\frac{d}{dx}\phi(x)$ i.e. $\frac{d^2y}{dx^2}$ and $\frac{k}{h}\to\frac{dy}{dx}$.

Hence, on taking limits, (3) becomes

$$(\beta-y)\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2-1=0,$$

i.e. $$\beta=y+\frac{1+(dy/dx)^2}{d^2y/dx^2}.$$

As $(\alpha,\ \beta)$ is a point on (1), we get also

$$\alpha=x-\frac{\frac{dy}{dx}\left\{1+(dy/dx)^2\right\}}{d^2y/dx^2}.$$

Sometimes these formulae are derived by showing geometrically that $\alpha=x-\rho\sin\psi$, $\beta=y+\rho\cos\psi$, the curve being drawn convex towards the x-axis, when ψ is positive. If, however, a curve concave to the x-axis is taken, the result would be wrong. The trouble is that the sign of ρ has not been defined. Hence the geometrical proof should be discarded.

We notice that $(\alpha-x)^2+(\beta-y)^2=\rho^2$

where $CP=\rho$ (see definition of §10.11). This ***equation represents the circle of curvature*** at the point $P\,(x,y)$.

Ex. Find the coordinates of the centre of curvature for the point (x,y) on the parabola $y^2=4ax$. [*Lucknow* **2008**]

Differentiating the given equation, we have

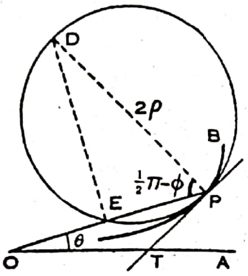
$$2y\,dy/dx=4a \quad \text{i.e.} \quad dy/dx=2a/y=a^{1/2}x^{-1/2}.$$

Therefore $d^2y/dx^2=-\frac{1}{2}a^{1/2}x^{-3/2}$.

Hence $$\alpha=x+\frac{2a^{1/2}x^{-1/2}(1+ax^{-1})}{a^{1/2}x^{-3/2}}=2a+3x.$$

Again $\beta = 2a^{1/2}x^{1/2} - \dfrac{1+ax^{-1}}{\frac{1}{2}a^{1/2}x^{-3/2}} = -2a^{-1/2}x^{3/2}$.

10.6. Chord of Curvature through the origin (or pole). Let BP be a curve and let PDE be the circle of curvature at P with C being its centre of curvature so that $CP = \rho$ and $DP = 2\rho$.



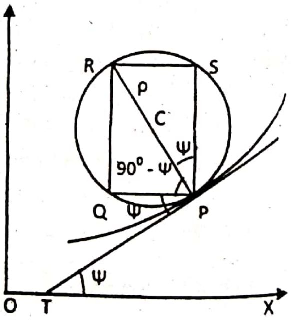
Let PE be the chord of curvature which passes through the origin (pole) and PT be the tangent at P. Then, the angle between the radius vector OP and the tangent PT is ϕ, i.e. $\angle OPT = \phi$.

Now, by geometry, the angle in a semicircle is a right angle. So, from the figure, in the right - angled triangle DPE, right - angled at E,

$$PE = DP \cos\angle DPE = 2\rho\cos\left(\tfrac{1}{2}\pi - \phi\right) = 2\rho\sin\phi.$$

Hence, PE, *the chord of curvature through the origin (pole), is equal to* $2\rho\sin\phi$.

10.61. Chord of Curvature parallel to the axes. Let PQ be the chord of curvature parallel to the x-axis and PS to the y-axis. Let the tangent PT at P make an angle ψ with the x-axis.



Since PQ is parallel to the x-axis, then, by geometry of parallel lines, $\angle QPT = \psi$ (alternate angle). So, from the figure, in the right - angled triangle PQR, right - angled at Q,

$$PQ = RP\cos\angle RPQ = 2\rho\cos\left(\tfrac{1}{2}\pi - \psi\right) = 2\rho\sin\psi.$$

Similarly, in the right- angled triangle PSR, right-angled at S,

$$PS = RP\cos\angle RPS = 2\rho\cos\psi.$$

Hence, *the chord of curvature parallel to the x-axis is* $2\rho\sin\psi$ *and that parallel to the y-axis is* $2\rho\cos\psi$.

Ex. 1. Show that the chord of curvature through the pole of the curve $r^n = a^n \cos n\theta$ is $2r/(n+1)$. [*Lucknow* 05, *Jabalpur* 07]

OR Find the chord of curvature through the pole of the curve $r^n = a^n \cos n\theta$. [*Rohilkhand* 05, *Avadh* 07]

We have $$r^n = a^n \cos n\theta \qquad (1)$$

Differentiating logarithmically, $\dfrac{n}{r}\cdot\dfrac{dr}{d\theta} = -n\cdot\dfrac{\sin n\theta}{\cos n\theta}$.

Therefore $\tan\phi = r\dfrac{d\theta}{dr} = -\cot n\theta = \tan\left(\tfrac{1}{2}\pi + n\theta\right)$.

This gives $\phi = \tfrac{1}{2}\pi + n\theta$ so that

$$\sin\phi = \sin\left(\tfrac{1}{2}\pi + n\theta\right) = \cos n\theta = r^n/a^n \qquad \text{[using (1)]}$$

Also $\rho = a^n r^{-n+1}/(n+1)$. (See Ex. 3. after § 10.35).

Hence the chord of curvature through the pole $= 2\rho\sin\phi$

$$= 2a^n r^{-n+1}(r^n/a^n)/(n+1) = 2r/(n+1).$$

Ex. 2. If C_x and C_y be the chords of curvature parallel to the axes of x and y at any point of the curve $y = a e^{x/a}$, prove that $1/C_x^2 + 1/C_y^2 = 1/2aC_x$. [*Agra* 07, *Rohilkhand* 07]

The angle between the tangent PT at P and the chord of curvature PQ parallel to the x-axis is ψ (Refer to figure of § 8.15).

Now $\tan\psi = dy/dx = y' = e^{x/a}$

and $y'' = (1/a)e^{x/a} = (1/a)\tan\psi$

Therefore $$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{a(1+\tan^2\psi)^{3/2}}{\tan\psi}$$

$$= a\sec^2\psi\ \operatorname{cosec}\psi. \qquad (1)$$

Hence C_x = chord of curvature parallel to the x-axis

$$= 2\rho\sin\psi = 2a\sec^2\psi \qquad \text{[using (1)]}$$

Also C_x = chord of curvature parallel to the y-axis

$$=2\rho\cos\psi=2a\sec\psi\operatorname{cosec}\psi. \qquad \text{[using (1)]}$$

Therefore $1/C_x^2+1/C_y^2=(1/4a^2)\{\cos^4\psi+\cos^2\psi\sin^2\psi\}$

$$=(1/4a^2)\cos^2\psi=1/2aC_x.$$

Ex. 3. In the curve $y=a\log\sec(x/a)$, prove that the chord of curvature parallel to the axis of y is of constant length.

[*Rohilkhand* 09, *Lucknow* 10, *Bilaspur* 10]

We have $y=a\log\sec(x/a)$

So
$$\frac{dy}{dx}=\frac{a\sec(x/a)\tan(x/a)}{a\sec(x/a)}=\tan(x/a) \qquad (1)$$

and $d^2y/dx^2=\frac{1}{2}\sec^2(x/a)$ so that

$$\rho=\frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2}=\frac{\{1+\tan^2(x/a)\}^{3/2}}{(1/a)\sec^2(x/a)}=a\sec(x/a). \qquad (2)$$

Let the tangent at a point P on the curve make an angle ψ with the x- axis. Then the chord of curvature parallel to the axis of y makes the same angle ψ with the normal to the curve at P.

So the length of the chord of curvature parallel to the axis of y

$$=2\rho\cos\psi \qquad (3)$$

Now, $\tan\psi=\dfrac{dy}{dx}=\tan x/a,$ [from (1)]

which gives $\psi=x/a.$

Therefore (3) becomes

$$2a\sec(x/a).\cos(x/a)=2a, \qquad \text{[from (2) and (3)]}$$

showing that this length is constant.

EXAMPLES

1. Find the centre of curvature of the following curves at the points indicated :

(i) $y=3x^3+2x^2-3$ at $(0,-3)$

(ii) $y=x^3-6x^2+3x+1$ at $(1,-1)$

2. In the parabola $x^2=4ay$, prove that the coordinates of the centre of curvature are $(-x^3/4a^2, 2a+3x^2/4a)$.

3. Show that the centre of curvature at the point determined by t on the ellipse $x=a\cos t$, $y=b\sin t$, is given by

$$x=\frac{a^2-b^2}{a}\cos^3 t,\quad y=-\frac{a^2-b^2}{b}\sin^3 t. \qquad \text{[\textit{Lucknow} 2007]}$$

4. Prove that the coordinates of the centre of curvature at any point (x, y) can be expressed in the form $x-\dfrac{dy}{d\psi}$ and $y+\dfrac{dx}{d\psi}$.

5. In the curve $y=c\cosh(x/c)$, show that the coordinates of the centre of curvature are

$$X=x-y\{(y^2/c^2)-1\}^{1/2},\ Y=2y. \qquad \text{[\textit{Lucknow} 2003]}$$

6. Show that $(x-\frac{3}{4}a)^2+(y-\frac{3}{4}a)^2=\frac{1}{2}a^2$ is the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$, at the point $(\frac{1}{4}a, \frac{1}{4}a)$.

7. Prove that the points on the curve $r=f(\theta)$, the circle of curvature at which passes through the origin, are given by the equation

$$f(\theta)+f''(\theta)=0. \qquad \text{[\textit{Garhwal} 2008]}$$

8. Prove that the equation of the circle of curvature of the parabola $y^2=4ax$ at $(am^2, 2am)$ is [*Raipur* 2011]

$$x^2+y^2-6am^2x-4ax+4am^3y-3a^2m^4=0.$$

9. Show that the chord of curvature, through the pole, of the equiangular spiral $r=ae^{m\theta}$ is $2r$.

10. Find the chord of curvature through the pole of the cardioid $r=a(1+\cos\theta)$. [*Lucknow* 02, *Indore* 03]

11. Show that the chord of curvature through the pole for the curve $p=f(r)$ is $2f(r)/f'(r)$.

12. Show that the length of the chord of curvature, parallel to the axis of y, at the origin, in the parabola $y=mx+x^2/a$, is $(1+m^2)a$.

13. Show that in any curve, the chord of curvature perpendicular to the radius vector is $2\rho(r^2-p^2)^{1/2}/r$.

[*Lucknow* 2003]

14. If C_r and C_θ be the chords of curvature of the curve $r=a(1+\cos\theta)$, through the pole and perpendicular to the radius vector, prove that $3(C_r^2+C_\theta^2)=8aC_r$. [*Rohilkhand* 2008]

ANSWERS

1. (i) $0, \ -2\frac{3}{4}$. (ii) $-36, \ -7\frac{1}{6}$. **10.** $4r/3$.

10.7. Concavity, Convexity and Point of Inflexion.

Let P be a given point on a curve and S a given straight line

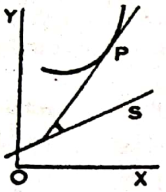


Fig. 1

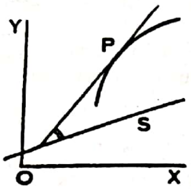
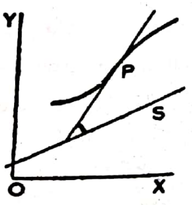


Fig. 2

which does not pass through P. Then the curve is said to be ***concave*** or ***convex*** at P with respect to S, according as a sufficiently small arc containing P lies entirely *within or without* the acute angle formed by S and the tangent to the curve at P.



Thus in Fig. 1, the curve at P is convex to S, and in Fig. 2, it is concave.

If the curve on one side of P is concave, and on the other side of P convex to any line S, the curve evidently crosses its tangent at P. Such a point is called a ***point of inflexion.***

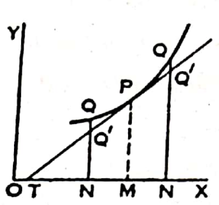
10.71. A test of concavity or convexity. We shall consider concavity and convexity with respect only to the axis of x.

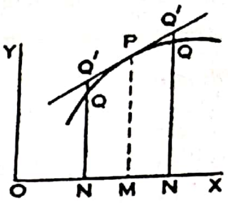
Let the equation to the curve be $y = f(x)$ and let P be the point $(x, \ y)$ and Q the point $(x + h, \ \ y + k)$.

Let the ordinate QN of Q meet the tangent to the curve at P in Q'.

Then the equation of the tangent is

$$Y - y = f'(x)(X - x).$$





Therefore it follows (by putting $X = x + h$) that

$$NQ' = y + f'(x)h.$$

Also, by Taylor's Theorem, if $0 < \theta < 1$,

$$NQ = f(x+h) = f(x) + h\,f'(x) + (1/2!)\,h^2 f''(x) + \dots$$
$$+ \{1/(n-1)!\}\,h^{n-1} f^{n-1}(x) + (1/n!)\,h^n f^n(x + \theta\,h).$$

Hence, by subtraction,

$$NQ - NQ' = h^2\{(1/2!)f''(x) + (1/3!)\,h\,f'''(x) + \dots$$
$$+ (1/n!)\,h^{n-2} f^n(x + \theta\,h)\}.$$

Now, if $f''(x)$ is not zero and we take h small enough, the sign of the right-hand side will be the same as that of $f''(x)$ whether h is positive or negative.

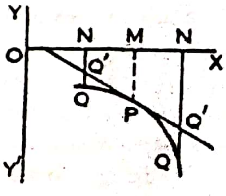
Hence the curve is convex at P to the axis of x if $f''(x)$ is positive, and concave if $f''(x)$ is negative.

We have drawn the above figures for the case when the curve is above the axis of x. If, however, it is below, NQ and NQ' are both negative, and

$$NQ - NQ' = -\{|NQ| - |NQ'|\}.$$

Hence in this case, the curve is convex at P to the axis of x if $|NQ| - |NQ'|$ be positive; i.e. if $NQ - NQ'$ be negative, i.e. if $f''(x)$ be negative.

Similarly the curve would be concave at P to the axis of x if $f''(x)$ be positive.



Both the cases are included in the statement:

A curve is convex or concave at P to the axis of x according as $y\dfrac{d^2y}{dx^2}$ is positive or negative at P.

In case the higher differential coefficients of $f(x)$ do not exist, we can stop the Taylor's expansion at the term which involves h^2. It is only necessary that $f''(x)$ should exist and be continuous at the point under consideration. We have

$$NQ = f(x+h) = f(x) + h f'(x) + (1/2!)h^2 f''(x+\theta h).$$

Hence, $\quad NQ - NQ' = (1/2!)h^2 f''(x+\theta h).$

Now, if $f''(x)$ is not zero, and we take h small enough, the sign of the right-hand side will, on account of the continuity at x of the second differential coefficient of $f(x)$, be the same as that of $f''(x)$. This will be so whether h is positive or negative.

Hence the preceding result will be true even if the third and the higher differential coefficients of $f(x)$ do not exist.

10.72. Test for Point of Inflexion. If, at P, $f''(x)$ is zero, but not $f'''(x)$, the above investigation shows that

$$NQ - NQ' = \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \ldots + \frac{h^n}{n!}f^n(x+\theta h).$$

For sufficiently small values of h the sign of the right-hand side is the same as that of $h^3 f'''(x)$ which changes sign when h changes sign. So the curve is concave to the axis of x on one side of P, and convex on the other. Hence ***there is a point of inflexion at P, if***

$$\frac{d^2y}{dx^2} = 0 \qquad \textit{but} \qquad d^3y/dx^3 \neq 0.$$

10.73. Points of undulation. If $f''(x) = f'''(x) = f^{iv}(x) = \ldots = f^{n-1}(x) = 0$, and $f^n(x) \neq 0$, it is easy to see from the value of $NQ - NQ'$ that there would be a point of inflexion if n is odd. If, however, n is even, the curve does not cross the tangent. Such a point (if n is greater than 2) is called a ***point of undulation.***

To the eye a point of undulation appears just like an ordinary point.

10.74. Concavity and Convexity with respect to the axis of y. By considering y as the independent variable, we can easily show that a curve, at a given point P on it, is convex or concave to the axis of y according as $x\,(d^2x/dy^2)$ is positive or negative at P; and that, if $d^2x/dy^2 = 0$ at P, but $d^3x/dy^3 \neq 0$, there is a point of inflexion at P.

This test is useful when the tangent at P is parallel to the axis of y.

10.75. Concavity and Convexity with respect to a point. Let P be a given point on a curve and let PT be the tangent at P to the curve. Then the curve is said to be concave or convex at P to a given point A, according as the curve in the immediate neighbourhood of P does, or does not, lie entirely on the same side of the tangent PT as A.

The investigation of § 10.71 gives us at once the following proposition :

A given curve is convex or concave at P to the foot of the ordinate of P, according as y y'' is positive or negative.

Ex. 1. Find the range of values of x in which the curve

$$y = 3x^5 - 40x^3 + 3x - 20$$

is concave upwards or downwards. Also find its points of inflexion.

Since $\quad y = 3x^5 - 40x^3 + 3x - 20 \qquad (1)$

$y' = 15x^4 - 120x^2 + 3,$

$y'' = 60x^3 - 240x = 60x(x^2-4) = 60x(x+2)(x-2)$

and $\quad y''' = 180x^2 - 240.$

Now $y'' = 0$ at $x = -2, 0$ and 2, while $y''' \neq 0$ at these points. Therefore, substituting these values of x in (1), we have points of inflexion at $(-2, 198), (0, -20), (2, -238)$.

Again, $y'' > 0$ for $[-2, 0]$, and y is negative for these values of x. So the curve is concave to the axis of x i.e. concave upwards in $[-2, 0]$ and $[-2, \infty[$.

Now y'' is positive for $[0, 2]$ and y is negative. So the curve is concave to the axis of x for these values of x; i.e. the curve is concave downwards in $]-\infty, -2]$ and $[0, 2]$.

Ex. 2. Show that the points of inflexion of the curve $y^2=(x-a)^2(x-b)$ lie on the line $3x+a=4b$.

[*Agra* 06, *Bilaspur* 08, *Kashi Vidyapeeth* 12]

Here $y=(x-a)(x-b)^{1/2}$,

$y'=(x-b)^{1/2}+\frac{1}{2}(x-a)/(x-b)^{1/2}$,

and $y''=\frac{1}{2}(x-b)^{-1/2}+\frac{1}{2}(x-b)^{-1/2}+\frac{1}{2}(-\frac{1}{2})(x-a)(x-b)^{-3/2}$

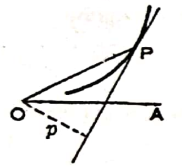
The points of inflexion are given by $y''=0$,

i.e. $(x-b)^{-1/2}-\frac{1}{4}(x-a)(x-b)^{-3/2}=0$,

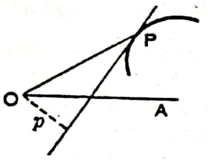
or $4(x-b)-(x-a)=0$, which gives $3x=4b-a$,

showing that they lie on the line $3x+a=4b$.

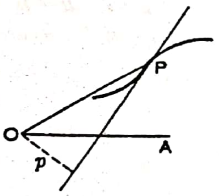
10.8. Concavity, Convexity and Point of Inflexion for Polar curves. It is evident from the figures that if, at any point P on the curve, p (the perpendicular from the pole on the tangent) increases as r increases, then the curve is concave at P to the pole, i.e. ***a curve is concave at P to the pole if dp / dr is positive there.***



Curve concave at *P* to *O*



Curve convex at *P* to *O*



Point of inflexion at *P*

Similarly, ***a curve is convex at P to the pole if dp / dr is negative there.***

If $dp/dr=0$ at P, positive for points on one side of P and negative for the points on the other side of P, *there must be a point of inflexion at P.*

$$\text{But } r\frac{dr}{dp}=\rho=\frac{\{r^2+(dr/d\theta)^2\}^{3/2}}{r^2+2(dr/d\theta)^2-r(d^2r/d\theta^2)}$$

$$\text{Hence}\quad \frac{dp}{dr}=\frac{r\{r^2+2(dr/d\theta)^2-r\,d^2r/d\theta^2\}}{\{r^2+(dr/d\theta)^2\}^{3/2}}.$$

For $\frac{dp}{dr}=0$, it follows that ***if* $r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}=0$ *at P, there is, in general, a point of inflexion at P.***

Ex. Find the points of inflexion of the curve $r^2\theta=a^2$.

The given curve is $r^2\theta=a^2$. (1)

Differentiating (1) w.r.t. θ, we have $r^2+2r\theta(dr/d\theta)=0$,

or $dr/d\theta=-r/2\theta=-r^3/2a^2$ [using (1)]

and $\frac{d^2r}{d\theta^2}=\frac{-1}{2a^2}\left(3r^2\frac{dr}{d\theta}\right)=\frac{3r^2}{2a^2}\cdot\frac{r^3}{2a^2}=\frac{3r^5}{4a^4}$.

At a point of inflexion, $r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}=0$.

or $r^2+2\left(\frac{-r^3}{2a^2}\right)^2-r\left(\frac{3r^5}{4a^4}\right)=0$ i.e. $r^2-\frac{r^6}{4a^4}=0$

or $r^4=4a^4$ which gives $r=\pm a\sqrt{2}$.

Substituting these values of r in (1), we have, $\theta=1/2$. Hence the points of inflexion are $(\pm a\sqrt{2},\frac{1}{2})$.

10.81. The sign of ρ. If in the Cartesian formula for ρ (§ 10.3), we always attach the positive sign to $(1+y'^2)^{3/2}$, ρ will be positive or negative according as y'' is positive or negative; i.e., according as the curve is concave upwards or convex upwards at (x, y). (See § 10.71 and consider separately the cases when y is positive and when y is negative.)

Similarly, § 10.8 shows that if we always attach the positive sign to $(r^2+r'^2)^{3/2}$ in the polar formula, ρ will be positive or negative according as the curve at (r,θ) is concave or convex to the origin. In the case of the pedal equation, an examination of the figures for the various possible cases (see § 10.8) shows at once that ρ is positive or negative at a given point on the curve according as the curve is concave or convex to the pole at that point.

10.9. Contact of curves. Let $y=\phi(x)$ and $y=\psi(x)$ be the equations to two curves.

Then, if $\phi(a)=\psi(a)$, the curves intersect at the point for which $x=a$. Let P be this point.

If $\phi(a)=\psi(a)$, and also $\phi'(a)=\psi'(a)$, the tangents to the two curves at P are the same; hence the two curves touch each other at P. We say that the curves have a ***contact of the first order*** at P, provided $\phi''(a)\neq\psi''(a)$.

If $\phi(a)=\psi(a)$, $\phi'(a)=\psi'(a)$, and also $\phi''(a)=\psi''(a)$, the curves are said to have a ***contact of the second order*** at P, provided $\phi'''(a)\neq\psi'''(a)$.

In general, if $\phi(a)=\psi(a), \phi'(a)=\psi'(a), \ldots, \phi^{n}(a)=\psi^{n}(a)$, and $\phi^{n+1}(a)\neq\psi^{n+1}(a)$, the curves are said to have a ***contact of the nth order*** at the point $x=a$.

EXAMPLES

1. Find the range of values of x for which the curve

$$y=x^4-6x^3+12x^2+5x+7$$

is concave upwards or downwards. Also find the points of inflexion.

[***Purvanchal*** **08,** ***Kashi Vidyapeeth*** **08**]

2. Show that $y=e^x$ is everywhere concave upwards whereas $y=\log x$ is everywhere concave downwards.

3. Examine $y=\cos x$ for concavity and convexity in the range $(0,\ 2\pi)$. [***Lucknow*** **05,** ***Jabalpur*** **09**]

Find the points of inflexion of the following curves:

4. $y=3x^4-4x^3+1$ [***Avadh*** **10,** ***Bilaspur*** **10,** ***Rohilkhand*** **10**]

5. $x=3y^4-4y^3+5$. 6. $y=(x-2)^6(x-3)^5$.

7. $y=e^{-x^2}$. [***Agra*** **2004**]

8. $x=(\log y)^3$. [***Kashi Vidyapeeth*** **09,** ***Bilaspur*** **09**]

9. Show that the curve $y(a^2+x^2)=a^2x$ has three points of inflexion and trace it. [***Lucknow*** **2008**]

10. Is there a point of inflexion at $x=0$ on the curve $y=x^4$?

11. Show that the curve $x^3+y^3=a^3$ has inflexions where it cuts the coordinate axes.

12. For the curve $(\theta^2-1)r=a\theta^2$, show that there is a point of inflexion at the point where $r=3a/2$

ANSWERS

1. Concave upwards in $]-\infty,1[$, $]2,\infty[$ and concave downwards in $[1,2]$. Pts. of inflexion at $(1,19)$ and $(2,33)$.

3. Concave in $[0,\frac{1}{2}\pi]$, $[\frac{3}{2}\pi,2\pi]$ and convex in $[\frac{1}{2}\pi,\frac{3}{2}\pi]$.

4. $(0,1)$, $(\frac{2}{3},\frac{11}{27})$. **5.** $(5,0)$, $(\frac{119}{27},\frac{2}{3})$.

6. Pts. of inflexion at $x=3,\frac{1}{11}(28\pm\sqrt{3})$.

7. $(\pm\frac{1}{\sqrt{2}},e^{-1/2})$. **8.** $(0,1),(8,e^2)$. **10.** No.

11

Singular Points, Curve Tracing

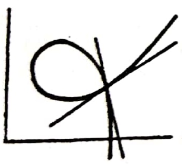
11.1. Multiple Points. A point of inflexion is a ***singular point*** (i.e. an unusual point) on the curve, for the tangent does not usually cross the curve, as it does at a point of inflexion. But there are other kinds of singularities. A point, for example, may be a multiple point.

A point through which more than one branches of a curve pass is called a ***multiple point*** on the curve.

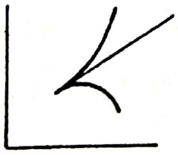
A point on a curve is called a ***double point*** if two branches of the curve pass through it; a ***triple point*** if three branches pass through it, and so on. If r branches pass through the point, it is called a ***multiple point of the rth order***.

If the two branches through a double point on a curve are real, and the tangents to them are not coincident, the double point is called a ***node***.

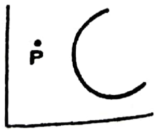
If the two tangents are coincident, the double point is called a ***cusp***.



Node



Cusp



Conjugate Point

Suppose there are no real points on the curve in the neighbourhood of a point P on the curve.

Then P is called a ***conjugate point*** (or an *isolated point*). The process of finding the tangent usually gives imaginary tangents at such a point, but sometimes the tangents come out real.

Remarks. (1) Conjugate points really arise from the fact that an imaginary expression $A+iB$ becomes real when $B=0$. Thus the imaginary "straight line".

$$x-2+i(y-4)=0 \qquad (1)$$

has one real point on it, viz., the point (2, 4).

This is a conjugate point on the straight line.

(2) The student must note that only those imaginary points whose coordinates satisfy (1) lie on it, not any and every imaginary point.

Thus, as substitution will immediately show, $(3+i, 3+i)$ lies on (1), but not $(4+i, 3+i)$. A similar statement is true for every curve having an imaginary branch. That is why the analytical process of finding the tangent at a conjugate point gives us a definite tangent, only the constants occurring in the tangent involve $\sqrt{-1}$.

(3) The student must not interpret "imaginary" as something which does not exist. The tangent to the curve $y=\sin(1/x)$ at $x=0$ does not exist. But it would be wrong to say that the tangent is imaginary there.

In fact the word "imaginary", is used in mathematics merely in the sense of "involving the square root of -1" and not in the sense of something which does not exist.

11.11. Species of Cusps. A cusp might be ***single*** or ***double***, according as the curve lies entirely on *one side* of the normal or on *both sides*.

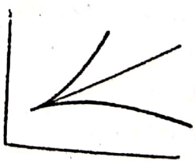


Fig. 1

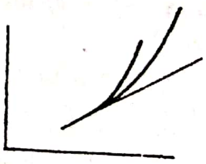


Fig. 2

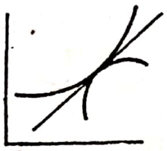


Fig. 3

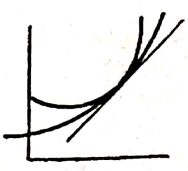


Fig. 4

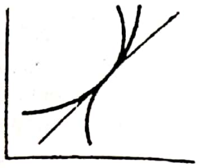


Fig. 5

Also, a single cusp might be of the ***first species*** or the ***second***, according as the two branches lie on *opposite* sides or on the *same* side of the tangent. Hence we have the following types of cusps:

Single cusp of the first species (Fig. 1).

Single cusp of the second species (Fig. 2)

Double cusp of the first species (Fig. 3).

Double cusp of the second species (Fig. 4).

Double cusp with change of species, or a point of ***osculinflexion*** (Fig. 5).

A cusp of the first species is also called a *keratoid cusp* (i.e. a *horn - like* cusp), and a cusp of the second species a *ramphoid* cusp (i.e. a *beak - like* cusp).

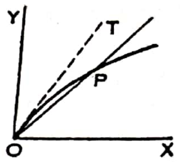
11.2. Tangents at the origin. In order to investigate the nature of a multiple point, it is necessary to find the tangent or tangents there. The following proposition is very helpful in this :

If a curve passes through the origin and is given by a rational, integral, algebraic equation, the equation to the tangent or tangents at the origin is obtained by equating to zero the terms of the lowest degree in the equation to the curve.

Let the equation to the curve be

$$a_1x + a_2y + b_1x^2 + b_2xy + b_3y^2 + c_1x^3 + \ldots = 0. \quad (1)$$

the constant term being absent, as the curve, by hypothesis, passes through the origin.



Let P be the point (x, y). Then the equation to OP is $Y = (y/x)X$.

The equation of the tangent at O is, therefore, $Y = \{\lim_{x\to 0}(y/x)\}X$. (2)

Let us exclude for the present the case when the tangent is the y-axis (viz. when $\lim_{x\to 0}(y/x) = \pm\infty$.)

Case I. Let $a_2 \neq 0$. Dividing (1) by x and taking limits as $x \to 0$, we get

$$a_1 + a_2\{\lim_{x\to 0}(y/x)\} = 0. \quad (3)$$

Substituting the value of $\lim_{x\to 0}(y/x)$ from this in (2), or what comes to the same thing, eliminating $\lim_{x\to 0}(y/x)$ between (2) and (3), we have

$$a_1X + a_2Y = 0;$$

or, since there is now no danger of confusion, writing x, y for X, Y, the tangent at the origin to (1) is

$$a_1x + a_2y = 0;$$

which could have been obtained by equating to zero the terms of the lowest degree in (1).

If $a_2 = 0$, then by (3), $a_1 = 0$, and we get the next case.

Case II. Let $a_1 = 0$ and $a_2 = 0$, but suppose that b_2 and b_3 are not both zero.

Divide (1) by x^2, and take limits. If $\lim_{x\to 0}(y/x)$ be denoted by m, we see that

$$b_1 + b_2m + b_3m^2 = 0 \quad (4)$$

which shows that there are, in general, two values of m, and therefore two tangents at the origin. Their equation is obtained by eliminating m between (4) and (2), and is

$$b_1x^2 + b_2xy + b_3y^2 = 0.$$

This equation could have been written down by equating to zero the terms of the lowest degree in the equation to the curve (viz. (1) with $a_1 = 0$, $a_2 = 0$).

If $b_2 = b_3 = 0$, then by (4), $b_1 = 0$.

Case III. If $a_1 = a_2 = b_1 = b_2 = b_3 = 0$, we can show similarly that the rule still holds; and so on.

If the tangent at the origin is the y-axis, we can easily see, by supposing the axis of x and y to be interchanged for an instant, that the rule is still true.

Hence by equating to zero the terms of the lowest degree, all the tangents will be obtained, including the y-axis if it is a tangent.

11.21. Nature of a cusp at the origin when the axis of x is a tangent. *If the origin is a cusp and the common tangent to the two branches of the curve is the axis of x, solve for y.*

The reality or otherwise of the roots for positive and negative values of x, also the fact of the roots being of the same or of opposite signs, will decide what type of cusp there is at the origin.

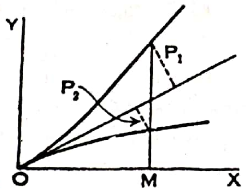
Since we want only the approximate shape of the curve in the immediate neighbourhood of the origin, the terms which are very small for small values of x, viz. terms involving high powers of x, may be neglected in deciding the nature of the cusp, provided that this does not reduce the equation merely to that of the tangent, or does not make the two branches of the curve coincide. Moreover, for the same reason and because there are only two branches of the curve through the origin (as we know from the equation to the tangents at the origin), terms involving powers of y above the second can also be neglected.

In this connection the student must remember that *for sufficiently small values of x, the largest term in an expression is the one involving the lowest power of x,* whatever its coefficient (provided, of course, that it is not zero).

11.22. Nature of a cusp at the origin. Other cases. If the tangent at the origin is the y-axis, we can proceed as in the last article, solving for x instead of y.

But if the tangent is neither the x-axis, nor the y-axis, but say, the line

$$aX + bY = 0.$$



Then the perpendicular p_1 on this line, from the point (x, y) on the curve is $\dfrac{ax + by}{\sqrt{a^2 + b^2}}$, which is proportional to $ax + by$. Let us put $P = ax + by$. If we eliminate y between this and the equation to the curve, i.e. if we substitute $(P - ax)/b$ for y in the equation to the curve, we get a relation between P and x. Solving for P, we shall be able to decide, from the sign and the reality of the values of P for positive and negative values of x, the nature of the cusp. Thus, if its values are of opposite signs (as in the figure) for a positive small x, the perpendiculars, from those points on the curve which have this value of x as abscissa, on the tangent $ax + by = 0$, are of opposite signs. Hence there is a cusp of the first species on the right; and so on.

11.23. Cusp at any point. In order to find the nature of the cusp at any point, it is usual *to transfer the origin to that point* and then apply the methods of the preceding articles.

Ex. 1. Find the nature of the origin on the curve

$$a^4 y^2 = x^4 (x^2 - a^2).$$

The origin lies on the curve; but $\quad y = \pm \dfrac{x^2}{a^2} \sqrt{x^2 - a^2},$

so that the values of y are imaginary, whether x is positive or negative, when x is small but $\neq 0$. Hence the origin is a conjugate point on the curve. Also

$$\frac{dy}{dx} = \pm \left\{ \frac{2x}{a^2} \sqrt{x^2 - a^2} + \frac{x^2}{a^2} \frac{x}{\sqrt{x^2 - a^2}} \right\}.$$

Hence $\dfrac{dy}{dx} = 0$ at the origin (0, 0). So the tangent at (0, 0) is

$$Y - 0 = 0(X - 0), \quad \text{i.e.} \quad Y = 0,$$

which is *real*, showing that the tangent may be real at a conjugate point.

[If we apply § 11.2, then also we get for the tangent at the origin, the equation $Y^2 = 0$ i.e. $Y = 0$].

Ex. 2. Examine the nature of the origin on the curve

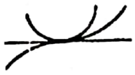
$$a^3y^2-2abx^2y+x^5+c^{-1}x^6=0.$$

Here the term of the lowest degree equated to zero gives $y^2=0$, showing that there are two coincident tangents ($y=0$) at the origin.

Solving for y, after neglecting x^6, we have

$$y=\frac{abx^2\pm\sqrt{a^2b^2x^4-a^3x^5}}{a^3}. \qquad (1)$$

When x is very small numerically, $a^2b^2x^4-a^3x^5$ has the same sign as $a^2b^2x^4$, which is positive whether x be positive or negative. Thus y is real for numerically small values of x (positive or negative). Hence we have a double cusp at the origin.



Also, when x is positive and small,

$$a^2b^2x^4-a^3x^5<a^2b^2x^4,$$

i.e. $$+\sqrt{a^2b^2x^4-a^3x^5}<abx^2,$$

Hence when x is positive and small, both the values of y are positive, i.e. on the right of the origin the curve is of the second species.

Again, when x is negative, and numerically small.

$$+\sqrt{a^2b^2x^4-a^3x^5}>abx^2,$$

because now $-a^3x^5$ is positive.

Thus, one value of y as given by (1) is now positive and the other negative. Hence there is a cusp of the first species on the left of the origin.

Thus there is an osculinflexion at the origin.

Ex. 3. Examine the nature of the origin on the curve

$$(2x+y)^2-6xy(2x+y)-7x^3=0.$$

The tangents at the origin obtained by equating to zero the lowest degree terms in the equation to the curve are

$(2x+y)^2=0$ i.e. two coincident tangents at the origin.

Putting $P=2x+y$, we have from the equation to the curve

$$P^2-6x(P-2x)P-7x^3=0 \qquad [\text{since } y=P-2x]$$

or $$(1-6x)P^2+12x^2P-7x^3=0.$$

Therefore $$P=\frac{-6x^2\pm\sqrt{36x^4+7x^3(1-6x)}}{1-6x}=\frac{-6x^2\pm\sqrt{7x^3-6x^4}}{1-6x}.$$

When x is negative the values of P are imaginary; and when x is positive, the values of P are real and of opposite signs (because, when x is small $x^{3/2}$ is greater than x^2).

Hence there is a single cusp of the first species at the origin.

11.3. Search for double points. In order to search for double points on a curve, transfer the origin to the point (h, k). In the transformed equation, ***the constant term and the terms of the first degree must be absent in order that the new origin may be a double point.*** This gives three equations, which are more than sufficient to determine h and k. Take any two of these and solve for h and k.

If the values obtained satisfy the remaining equation, the point (h, k) will be a double point. If no set of values of h and k can be found to satisfy all the three equations, there is no double point anywhere on the curve.

Ex. 1. Determine the position and character of the double points on the curve $(x-2)^2=y(y-1)^2$. [*Gorakhpur* 05, *Rohilkhand* 12]

Transferring the origin to (h, k) the equation becomes

$$(x+h-2)^2-(y+k)(y+k-1)^2=0$$

or $$(h-2)^2-2x(h-2)-k(k-1)^2-y\{2k(k-1)+(k-1)^2\}$$
$$+\text{terms of higher degree}=0.$$

Hence, for a double point

$$(h-2)^2-k(k-1)^2=0 \qquad (1)$$

$$h-2=0 \qquad (2)$$

and $$2k(k-1)+(k-1)^2=0. \qquad (3)$$

Eqn. (2) gives $h=2$, and (3) gives $k=1$ or $1/3$.

By substitution in the remaining equation, viz. eqn. (1), we see that $h=2, k=1$, satisfy it, but not $h=2, k=\frac{1}{3}$. Hence there is a double point only at $(2, 1)$. Transferring the origin to it, the equation becomes

$$x^2=(y+1)y^2$$

or $$x=\pm y\sqrt{y+1}. \qquad (2)$$

When y is positive and small, x has two real values, one positive and the other negative. When y is negative but numerically small, the same is still true.

The tangents at the new origin obtained by equating to zero the lowest degree terms in (2) are $x^2 = y^2$ or $x = \pm y$ i.e. real and distinct.

Hence there is a node at the point (2, 1) on the given curve.

Ex. 2. Find the nature and position of the multiple points on the curve $x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0$. [*Kumaon* 2011]

Also find the tangents at the multiple point. [*Delhi* 2002]

Transferring the origin to (h, k), the equation of the curve becomes

$$(x+h)^4 - 2(y+k)^3 - 3(y+k)^2 - 2(x+h)^2 + 1 = 0. \qquad (1)$$

or $$(h^4 - 2k^3 - 3k^2 - 2h^2 + 1) + x(4h^3 - 4h) + y(-6k^2 - 6k)$$
$$+ \text{terms of higher degree} = 0$$

Hence, for a multiple point

$$h^4 - 2k^3 - 3k^2 - 2h^2 + 1 = 0 \qquad (2)$$

$$4h^3 - 4h = 0 \quad (3) \quad \text{and} \quad -6k^2 - 6k = 0 \qquad (4)$$

Equation (3) gives $h = 0, 1$ or -1, and (4) gives $k = 0$ or -1.

Of the six points given by these h, k, only $(0, -1)$, $(1, 0)$ and $(-1, 0)$ satisfy equation (2). So these are the required multiple points. The lowest degree terms in (1) are then

$$6h^2x^2 - 6ky^2 - 3y^2 - 2x^2 = 0 \quad \text{or} \quad (6h^2 - 2)x^2 = (6k + 3)y^2 \qquad (5)$$

This gives the tangents at the multiple points.

At $(0, -1)$ i.e $h = 0, k = -1$, this gives

$$2x^2 = 3y^2 \quad \text{or} \quad y = \pm\sqrt{(2/3)}\,x.$$

At $(1, 0)$ i.e. $h = 1, k = 0$, this gives

$$4x^2 = 3y^2 \quad \text{or} \quad y = \pm\sqrt{(4/3)}\,x.$$

At $(-1, 0)$, i.e. $h = -1, k = 0$, (5) gives again $4x^2 = 3y^2$.

So the tangents at $(0, -1)$ are $y + 1 = \pm\sqrt{(2/3)}x$, at $(1, 0)$ are $y = \pm\frac{2}{\sqrt{3}}(x-1)$, and at $(-1, 0)$ are $y = \pm\frac{2}{\sqrt{3}}(x+1)$, all being real and distinct. Hence the three multiple points on the curves are nodes.

11.4. A necessary condition for the existence of double points. If $f(x, y)$ is an expression involving x and y, and the equation to a curve is $f(x, y) = 0$, then on transferring the origin to (h, k), the equation will become

$$f(x + h, y + k) = 0$$

It will be shown in the next chapter that, if $\frac{\partial f}{\partial x}$ means the differential coefficient of $f(x, y)$ when y is regarded as a constant, $\frac{\partial f}{\partial y}$ the differential coefficient when x is regarded as a constant and $\frac{\partial^2 f}{\partial x^2}$ means $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ and so on, then by Taylor's theorem for functions of two variables, we have

$$f(x+h, y+k) = f(h,k) + x\left(\frac{\partial f}{\partial x}\right)_{x=h,\,y=k} + y\left(\frac{\partial f}{\partial y}\right)_{x=h,\,y=k}$$
$$+\frac{1}{2}\left\{x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=h,\,y=k} + 2xy\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{x=h,\,y=k} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=h,\,y=k}\right\}$$
$$+ \text{terms of higher degrees.} \qquad (1)$$

Hence, by § 11.3, we see that at a double point we must have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad f = 0,$$

which is a necessary condition for the existence of double points.

If these conditions are satisfied, the tangents at the new origin are given by equating the second degree terms in (1) to zero, so that

$$x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=h,\,y=k} + 2xy\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{x=h,\,y=k} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=h,\,y=k} = 0.$$

The tangents at the new origin given by the above quadratic equation will be distinct, coincident or imaginary according as the discrminant is $>, =$ or < 0.

Hence, by § 11.1, in general, ***a double point will be a node, cusp, or conjugate point according as***

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 >, =, \text{ or } < \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right).$$

Ex. 1. Find the double points of the curve $x^3 + y^3 = 3axy$.

[***Agra*** **06,** ***Avadh*** **08,** ***Kumaon*** **08, 09**]

Let $\qquad f(x, y) \equiv x^3 + y^3 - 3axy = 0.$ (1)

Then $\qquad \partial f/\partial x = 3x^2 - 3ay$ and $\partial f/\partial y = 3y^2 - 3ax.$ (2)

We know that at a double point, $f = 0$, $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$.

Now $\partial f/\partial x = 0$ gives $x^2 = ay$ and $\partial f/\partial y = 0$ gives $y^2 = ax$.

Solving them, we get $\quad x^4 = a^2y^2 = a^3x$ or $x(x^3 - a^3) = 0$.

This gives $x = 0, a$ and hence $y = 0, \pm a$. (Taking only real values)

Thus the possible double points are (0, 0), $(0, a)$, $(0, -a)$, $(a, 0)$, (a, a) and $(a, -a)$ of which only (0, 0) satisfies (1).

Therefore (0, 0) is the only double point.

To find the tangents to the given curve at the origin (0, 0), we shall equate to zero the lowest degree terms in (1). This gives

$xy = 0$ i.e. $x = 0$, $y = 0$ as the tangents at (0, 0) which are real and distinct. Hence the origin (0, 0) is a node.

Ex. 2. Locate the double points of the curve

$y(y-6) = x^2(x-2)^3 - 9$ and ascertain their nature.

[***Gorakhpur*** **03,** ***Agra*** **04,** ***Delhi*** **08,** ***Rohilkhand*** **08,09,** ***Lucknow*** **04, 12**]

Let $\qquad f(x, y) \equiv x^2(x-2)^3 - y(y-6) - 9 = 0$ (1)

Then $\qquad \partial f/\partial x = 2x(x-2)^3 + 3x^2(x-2)^2 = x(x-2)^2(5x-4)$ (2)

$\qquad \partial f/\partial y = -(y-6) - y = 6 - 2y$ (3)

Now $\partial f/\partial x = 0$ gives $x = 0, 2, \frac{4}{5}$ and $\partial f/\partial y = 0$ gives $y = 3$.

So the possible double points are (0, 3), (2, 3) and $(\frac{4}{5}, 3)$ but since $(\frac{4}{5}, 3)$ does not satisfy (1) so there are only two double points (0, 3) and (2, 3). From (2) and (3), we have

$$\frac{\partial^2 f}{\partial x^2} = (x-2)^2(5x-4) + 2x(x-2)(5x-4) + 5x(x-2)^2$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = 0 \text{ and } \frac{\partial^2 f}{\partial y^2} = -2$$

Hence at (0, 3) $\qquad \dfrac{\partial^2 f}{\partial x^2} = -16,\ \dfrac{\partial^2 f}{\partial x\,\partial y} = 0,\ \dfrac{\partial^2 f}{\partial y^2} = -2$

Therefore $\qquad \left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right) < \left(\dfrac{\partial^2 f}{\partial x^2}\right)\left(\dfrac{\partial^2 f}{\partial y^2}\right)$

so (0, 3) is a conjugate point.

At (2, 3) $\qquad \dfrac{\partial^2 f}{\partial x^2} = 0,\ \dfrac{\partial^2 f}{\partial x\,\partial y} = 0,\ \dfrac{\partial^2 f}{\partial y^2} = -2$

Therefore $\qquad \left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right) = \left(\dfrac{\partial^2 f}{\partial x^2}\right)\left(\dfrac{\partial^2 f}{\partial y^2}\right)$

Hence, the point (2, 3) is a cusp.

To determine the nature of the cusp, transfer the origin to the point (2, 3). Then equation (1) becomes

$$(y+3)(y-3) = x^3(x+2)^2 - 9 \text{ or } y^2 = x^3(x+2)^2.$$

So, the tangents at the new origin are $y^2 = 0$ showing that the x-axis is the common tangent

Solving for y, we get $y = \pm(x+2)\sqrt{x^3}$.

For $x < 0$, y is imaginary.

For $x > 0$, y has a positive and a negative value.

Hence we have a single cusp of the first kind at (2, 3).

EXAMPLES

1. Write down the equations to the tangents at the origin for the following curves :

(i) $x^4 + 3x^3y + 2xy - y^2 = 0.$ (ii) $x^3 + 3xy + 7x^2 = 0.$

2. Show that the curve $y^3 = (x-a)^2(2x-a)$ has a single cusp of the first kind at the point $(a, 0)$. [***Lucknow*** **2004**]

3. Show that the curve $(xy+1)^2 + (x-1)^3(x-2) = 0$ has a single cusp of the first kind at the point $(1, -1)$.

4. Show that the cardioid $r = a(1 + \cos\theta)$ has a cusp at the origin.

5. Find the nature of the origin on the following curves :

(i) $x^4 + y^3 + 2x^2 + 3y^2 = 0$. [*Meerut* 2007]

(ii) $x^4 - ax^2y + axy^2 + a^2y^2 = 0$. [*Rohilkhand* 11, *Kumaon* 12]

(iii) $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$. [*Avadh* 2007]

(iv) $(x^2 - a^2)^2 (y^2 - b^2) = a^2b^2$.

(v) $x^3 + y^3 = 3xy$. [*Meerut* 05, *Agra* 10]

6. Prove that the curve $ay^2 = (x-a)^2 (x-b)$ has, at $x = a$, a conjugate point if $a < b$, a node if $a > b$ and a cusp if $a = b$.

Apply maclaurin-theorem

7. Show that the curve $y^2 = bx \tan (x/a)$ has a node or a conjugate point at the origin according as a and b have like or unlike signs.

[Hint : Expand tan (x/a) and retain only the first few terms.]

8. For the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$, show that the origin is a node and that the nodal tangents bisect the angles made by the axes.

9. Determine the position and nature of the double points on

(i) $y^2 = (x-2)^2(x-1)$. [*Kumaon* 2010]

(ii) $x^4 - 4y^3 - 12y^2 - 8x^2 + 16 = 0$.

(iii) $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$. [*Delhi* 03, *Rajasthan* 03]

(iv) $x^3 + x^2 + y^2 - x - 4y + 3 = 0$. [*Garhwal* 2010]

(v) $y^2 = bx \sin (x/a)$. [*Kumaon* 2013]

10. Show that the points of intersection of the curve

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

with the axes are cusps of the first species.

ANSWERS

1. (i) $2xy - y^2 = 0$; (ii) $3xy + 7x^2 = 0$.

5. (i) Conjugate point. (ii) Conjugate point.

(iii) A cusp of the second kind at the origin.

(iv) Conjugate point. (v) Node.

9. (i) Node at $(2, 0)$.

(ii) Node at each double point $(-2, 0)$, $(0, -2)$ and $(2, 0)$.

(iii) Node at $(3, 2)$. (iv) Node at $(-1, 2)$.

(v) A node or a conjugate point at the origin according as a and b have like or unlike signs.

11.5. Curve Tracing. Cartesian Equations. The object of Curve Tracing is to find the approximate shape of a curve without the labour of plotting a large number of points.

If the Cartesian Equation is given, the student will find that he can invariably *solve it either for y, or for x, or for r* (in terms of θ in the last case), otherwise the curve will be too difficult for him to trace.

Only curves in which we can solve for y need be considered here because, if the equation cannot be solved for y, but can be solved for x, we have only to regard y as the independent variable. If the equation can be solved for r, the rules for tracing polar curves will apply.

11.51. Procedure .

1. SYMMETRY. Notice if the curve is symmetrical about any line, by applying the following rules, whose truth is evident :

(i) If the powers of y which occur in the equation are all even, the curve is ***symmetrical about the axis of x.***

(ii) If the powers of x are all even, the curve is ***symmetrical about the axis of y.*** A curve might, of course, be symmetrical about both axes.

(iii) If x and y can be interchanged without altering the equation, the curve is ***symmetrical about the line*** $y = x$.

(iv) If on changing the signs of x and y both, the equation to the curve is not altered, the curve after being turned through two right angles will coincide with its old trace. (This is generally denoted by saying that there is ***symmetry in opposite quadrants.***)

2. AT THE ORIGIN. Notice if the curve passes through the origin (i.e. the constant term will be zero). If it does, write down the equation of the tangent(s) there. If the origin is a singular point, find its nature. Remember that the higher powers of x in the expression for y can be neglected when tracing the curve for (numerically) very small values of x.

3. SOLVE FOR y. Solve for y (which, by supposition, is possible). Choose any convenient value of x for which y is finite, and if possible zero (generally $x = 0$ is convenient). Consider how y will vary as x increases and then tends to infinity, paying particular attention to those values of x for which $y = 0$, or $\rightarrow \infty$.

If the curve is symmetrical about the x-axis, or if there is symmetry in opposite quadrants, only positive values of y need be considered. The curve for negative values of y can be drawn from symmetry.

4. CONSIDER ALL VALUES OF x. Starting from the chosen value of x, repeat the above procedure as x decreases and then $\rightarrow -\infty$.

Of course, if the curve is symmetrical about the y-axis, it can be drawn for negative values of x by symmetry, so such values of x need not be considered afresh.

5. IMAGINARY VALUES OF y. In the above procedure, if y is found to be imaginary for a certain range of values of x, say for values of x between a and b, it would mean that the curve does not exist in the region bounded by the lines $x = a$, $x = b$.

6. ASYMPTOTES. If the procedure of paragraphs 3 and 4 shows that the curve extends to infinity, and there is approximately a linear relation between x and y for numerically large values of x, then there is an oblique asymptote. This should now be found, and also, if necessary, it should be investigated on which side of it the curve lies.

NOTE. For numerically very large x and y, only the highest powers of these may be retained to find the approximate shape of the curve.

The presence of asymptotes parallel to the axes and their positions will become evident by the procedure of paragraphs 3 and 4.

7. SPECIAL POINTS. Find the coordinates of a few points on the curve if it appears necessary.

For example, if y is 0 at $x = a$, and again at $x = b$, and is positive for the intermediate values, it might be desirable to find the maximum (greatest) value of y between a and b. At the point for which y is maximum, the tangent (as is evident from geometry) will be horizontal and so dy/dx will be zero. Hence this point can be easily found (the subject is treated in detail in Chapter XV). Even if the maximum value is not found, it would be desirable to find the value of y when x is equal to, say, $\frac{1}{2}(a + b)$.

8. POINTS OF INFLEXION. If the curve as traced appears to possess a point of inflexion, that point can be more accurately located by putting d^2y/dx^2 or d^2x/dy^2 equal to zero and solving the equation thus obtained.

9. *The student should remember that merely a knowledge of symmetry, asymptotes, tangents at the origin, points of inflexion, double points, and the coordinates of a few other points will never enable him to trace a curve. His difficulties regarding curve-tracing will vanish only if he realises that we have solved for y and expressed it as a function of x whose values can be easily found for every value of x, and that his task is to save time by picking out the most important values of x (say those at which y is a minimum or a maximum, or is zero or infinity, or just begins to be imaginary or ceases to be so); then, by noticing how y varies (i.e. increases or diminishes) as x is made to vary continuously from* $-\infty$ *to* $+\infty$, *the curve is easily traced.*

We need not begin from $-\infty$ (if that is inconvenient) provided later we consider the remaining values also of x.

NOTE. 1. An equation of the second degree in x and y gives merely one of the conic sections, and so can be traced.

2. If the equation to a curve is given in a ***parametric form***, $x=\phi(t)$, $y=\psi(t)$, the parameter t can in some cases be easily eliminated and the curve traced. But generally it is more convenient to give to t a series of values and plot the corresponding values of x and y, noting how x and y would behave (e.g., increase or decrease) for the intermediate values.

Ex. 1. Trace the curve $y^2(a+x)=x^2(a-x), a>0.$

[*Kumaon* 09, *Kanpur* 10, *Lucknow* 09, 11, *Bundelkhand* 11, *Raipur* 13]

OR Trace the curve $x(x^2+y^2)=a(x^2-y^2)$. **[*Agra* 04, *Delhi* 08]**

(i) This curve is symmetrical about the axis of x.

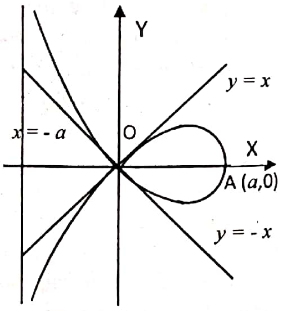
(ii) The curve passes through the origin. The tangents there are given by $ay^2=ax^2$ i.e. $y=\pm x$, which represents two real and distinct straight lines. Hence we may expect a node at the origin.

(iii) When the curve intersects the x-axis, then $y=0$ so $x^2(a-x)=0$ or $x=0, a$. Thus the curve intersects the x-axis at $(0, 0)$ and $(a, 0)$.

Again the curve intersects the y-axis, when $x=0$ so $y=0$ or at $(0, 0)$.

(iv) Solving for y, the equation of the curve gives

$$y=\pm x\sqrt{\frac{a-x}{a+x}}. \qquad (1)$$



Let us first consider only positive values of x.

If $x=0$, then $y=0$. When $0<x<a$, y is real. We notice also that as $\frac{a-x}{a+x}<\frac{a}{a}<1$, then from (1), $y<x$. Hence the curve exists and lies below the tangent $y=x$ for $0<x<a$ in the first quadrant.

As x goes on increasing, y next becomes zero at $x=a$.

When $x>a$, the expression under the radical sign in (1) is negative and so y is imaginary. Thus the curve does not exist in the region $x>a$.

(v) If $-a<x<0$, (1) shows that y is real. Also, for values of x under consideration, $a-x>a+x$. Hence from (1), $y>x$.

This shows that the curve lies above the tangent $y=-x$ in the second quadrant for the region $-a<x<0$.

Also as x decreases from 0 to $-a$, from (1), we see that y increases from 0 to $+\infty$ for positive values of y in the second quadrant.

(vi) When $x<-a$, the quantity under the radical in (1) is negative and so y is imaginary. Thus the curve does not exist in the region $x<-a$.

(vii) $x=-a$ is an asymptote of the curve since as $x\to -a$, $y\to\infty$.

(vii) To find the tangent at the new origin, we shall transfer the origin to $(a, 0)$, so the equation to the curve will become

$$y^2(2a+x)=(x+a)^2[a-(x+a)]$$

or

$$y^2(2a+x)=-x(x^2+2ax+a^2). \qquad (2)$$

Then the tangent at the new origin will be $x=0$, (the new y-axis) obtained by equating to zero the terms of the lowest degree in (2). So the tangent at the new origin is a line parallel to the y-axis through $(a, 0)$.

Therefore, taking symmetry into account, the complete curve must be as shown in the figure.

Ex. 2. Trace the curve $y^2(2a-x)=x^3$.

[*Delhi* 03, 06, *Rohilkhand* 06, *Bundelkhand* 04, 08, *Avadh* 03, 05, 08, *Garhwal* 09, *Bilaspur* 09, *Jabalpur* 09, *Agra* 06, 10, *Lucknow* 12, *Kashi Vidyapeeth* 12, *Raipur* 09, 11, 14]

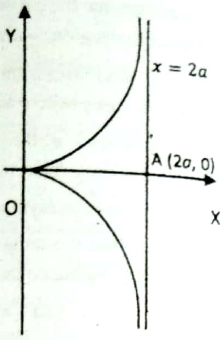
(i) The curve is symmetrical about the x-axis since its equation has even powers of y.

(ii) The curve passes through the origin, the tangents there being $y^2=0$, which represents two coincident straight lines. Hence there is a cusp at the origin.

(iii) The curve intersects the coordinate axes only at the origin.

(iv) The equation to the curve is a quadratic in y, and can be written as $y=x\sqrt{\frac{x}{2a-x}}$ considering only the positive value.

When $x<0$ or $x>2a$, y is imaginary which implies that the curve does not exist in the region to the left of the y-axis or the right of the line $x=2a$.



If $x = 0$, then $y = 0$. When $0<x<2a$, for the region above the x-axis, as x increases from 0 to $2a$, y also increases from 0 to ∞ and similarly, by symmetry, for the region below the x-axis.

(v) The asymptote parallel to the y-axis is $x=2a$, obtained by equating to zero the coefficient of the highest power of y, i.e. $x-2a=0$.

To find any other asymptote, put $x=1, y=m$ in the highest (i.e. third) degree terms of the equation to the curve and equate to zero. This gives $m^2+1=0$ whose roots are imaginary, so $x=2a$ is the only real asymptote of the curve.

Hence the shape of the curve is as shown.

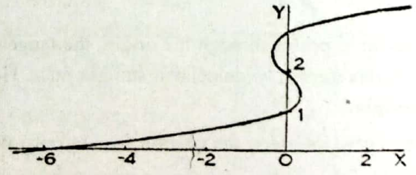
Ex. 3. Trace the curve $x=(y-1)(y-2)(y-3)$.

[*Kanpur* 09, *Kashi Vidyapeeth* 10]

(i) The curve is not symmetrical about the axes or about $x=y$.

(ii) It does not pass through the origin.

(iii) It is difficult to solve it for y. But it is already solved for x. Hence we take y as the independent variable.



When $y=0, x=-6$.

When $y=1$, $x=0$. Between $y=0$ and $y=1$, x is negative, as then all the three factors are negative.

When y lies between 1 and 2, x is positive as one factor is positive and two are negative. x next becomes zero at $y=2$.

Between $y=2$ and $y=3$, x is negative, x next becomes zero at $y=3$.

When $y>3$, x is positive. As $y\to\infty$, $x\to\infty$. For very large values of y, x is approximately equal to y^3. Hence there is no linear asymptote for this branch.

(iv) When y is negative, x is negative. As $y\to-\infty$, $x\to-\infty$. Again there is no linear asymptote for this branch also.

(v) When $y=1\frac{1}{2}, x=\frac{3}{8}$; when $y=2\frac{1}{2}, x=-\frac{3}{8}$.

Hence the shape of the curve is as shown.

[If we like we can also find where the tangent is parallel to the y-axis. At these points $dx/dy=0$,

i.e. $$(y-1)(y-2)+(y-1)(y-3)+(y-2)(y-3)=0,$$

or $$3y^2-12y+11=0$$

i.e. the tangent is parallel to the y-axis where

$$y=\frac{6\pm\sqrt{36-33}}{3}=2\pm\sqrt{3}/3=26 \text{ and } 1.4, \text{ nearly.}$$

We can now find the values of x for these values of y, and thus find the shape of the curve a little more exactly.]

Ex. 4. Trace the curve $y^2=\dfrac{x(x-a)(x-2a)}{x+3a}$.

(i) The curve is symmetrical about the axis of x.

(ii) The curve passes through the origin, the tangent there being $x=0$.

(iii) The equation is already solved for y. Considering only positive values of y, we see that :

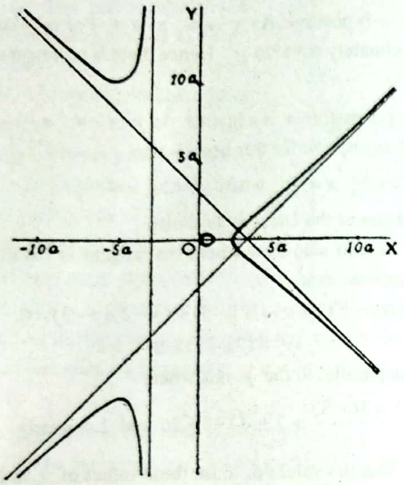
When $x=0$, $y=0$.

When $x>0$, but small, y is real.

When $x=a$, $y=0$ again [When $x=\frac{1}{2}a$, $y=\frac{1}{2}a$ nearly.]

Between $x=a$ and $x=2a$, y^2 is negative and so y is imaginary.

When $x=2a$, $y=0$. When $x>2a$, y is real. The tangent at $(2a, 0)$ is (as we can see mentally by supposing the origin to be transferred to it) parallel to the y-axis. [When $x=3a$, $y=a$; when $x=4a, y=1.8a$]. As $x\to\infty$, $y\to\infty$.



When x and y are large, we can write the equation as

$$\frac{y^2}{x^2}=\frac{[1-(a/x)][1-(2a/x)]}{[1+(3a/x)]}=1-\frac{6a}{x}+\frac{20a^2}{x^2}+\dots.$$

Hence $$\frac{y}{x}=\pm\left(1-\frac{3a}{x}+\frac{11a^2}{2x^2}+\dots\right), \tag{1}$$

or $y=x-3a+\dfrac{11a^2}{2x}+\dots.$ considering only positive values of y/x.

Hence $y=x-3a$ is an asymptote, and in the first quadrant the curve lies above the asymptote (§ 9.51).

(iv) When x is negative and numerically less than $3a$, y^2 is negative and so the curve does not exist between $x=0$ and $x=-3a$.

When $x=-3a$, from the left $y^2\to+\infty$. In fact $x=-3a$ is an asymptote.

As $x\to-\infty$, $y^2\to+\infty$. For numerically large values of x and y, we have (taking the negative value of y/x in (1) above)

$$y=-x+3a-\frac{11}{2}\frac{a^2}{x}+\dots.$$

Hence the asymptote is $y=-x+3a$, and in the 2nd quadrant also the curve is above the asymptote.

Thus in the second quadrant y is very large near $x=-3a$ and again very large for large negative values of x.

[For the intermediate values we can do one of two things. Either find y for a few values of x, say $x=-4a, -5a$ and $-6a$; or find where the tangent is horizontal (i.e. y has the least value: see Chapter XV). For the latter we shall have to put $dy/dx=0$ and solve. It is easy to see that we shall get thus a cubic in x, which might be difficult to solve. Hence we adopt the former alternative.]

When $x=-4a$, $y^2=120a^2$, $x=-5a$, $y^2=105a^2$,
$x=-6a$, $y^2=112a^2$, $x=-7a$, $y^2=126a^2$.

It is easy to see that y is a minimum somewhere near $x=-5a$; also, that as we take still larger negative values of x, y^2 would be larger.

Taking all the above facts into consideration, we see that the complete curve must be as shown in the figure.

[The curve appears to have two points of inflexion. By equating d^2y/dx^2 to zero and solving for x, we can locate these if we like.]

Ex. 5. Trace the curve $x^5+y^5-5a^2x^2y=0$.

Here the equation is a quintic in y which cannot be solved. Neither can we solve the given equation for x. But, transforming the equation to polars by putting $x=r\cos\theta$ and $y=r\sin\theta$, we have at once

$$r^2=\frac{5a^2\cos^2\theta\sin\theta}{\cos^5\theta+\sin^5\theta},$$

which can be easily traced by the method for polar curves (see § 11.6).

EXAMPLES

Trace the following curves :

1. (i) $y=x^3$. **[*Sagar* 03, *Bilaspur* 07]**
   (ii) $y^2=x^3$. **[*Bundelkhand* 2006]**

2. $y=x(x^2-1)$.

3. $y=x(x^2+1)$.

~~4.~~ $y=(x-2)(x+1)^2$.

5. $x^2y=x+1$.

6. $x^3y=x+1$.

7. $y=x^3-7x+6$.

8. $y=x^4-3x^2+2$. **[*Jabalpur* 06, *Agra* 08]**

~~9.~~ $y(x^2+4a^2)=8a^3$.

8,9,4

10. $y(x^2-1)=x^2+1$. [*Garhwal* 08, *Agra* 11, *Kashi Vidyapeeth* 11]

11. $x^3+y^3=a^2x$ [*Meerut* 06, 10, *Kanpur* 08]

12. (i) $ay^2=x^2(a-x)$. [*Jabalpur* 2004]

(ii) $ay^2=x^2(x-a)$.

13. $9ay^2=x(x-3a)^2$.

14. $a^{3/2}y=(x-a)^2\sqrt{x-b}$, $a>b$.

15. $y^2(x+a)=(x-a)^3$. [*Meerut* 04, *Bundelkhand* 12]

16. $(x^2-a^2)(y^2-b^2)=a^2b^2$.

17. $y^2(x^2+y^2)+a^2(x^2-y^2)=0$. [*Lucknow* 2003]

18. $y^2=(x-2)^2(x-5)$.

19. $x^2y^2=(1+y)^2(4-y^2)$.

20. $x(x-2a)y^2=a^2(x-a)(x-3a)$. [*Lucknow* 2006]

21. $x(y-x)^2=ay^2$.

22. $a^5y^2+2a^3x^3y+x^7=0$.

23. $y=x^2/a+\sqrt{a^2-x^2}$.

24. $(x-1)^3(x+1)y^3=x^4$.

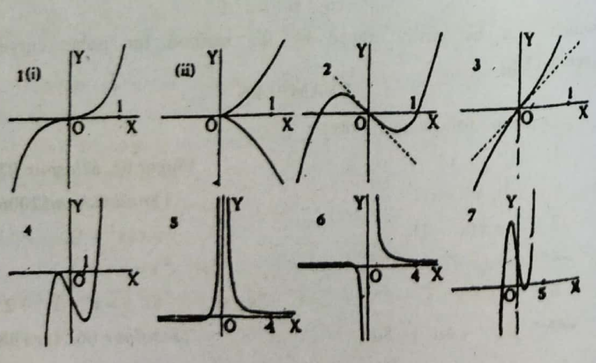
25. Show that the curve $(a^2+x^2)y=a^2x$ has three points of inflexion, and trace it. [*Lucknow* 2008]

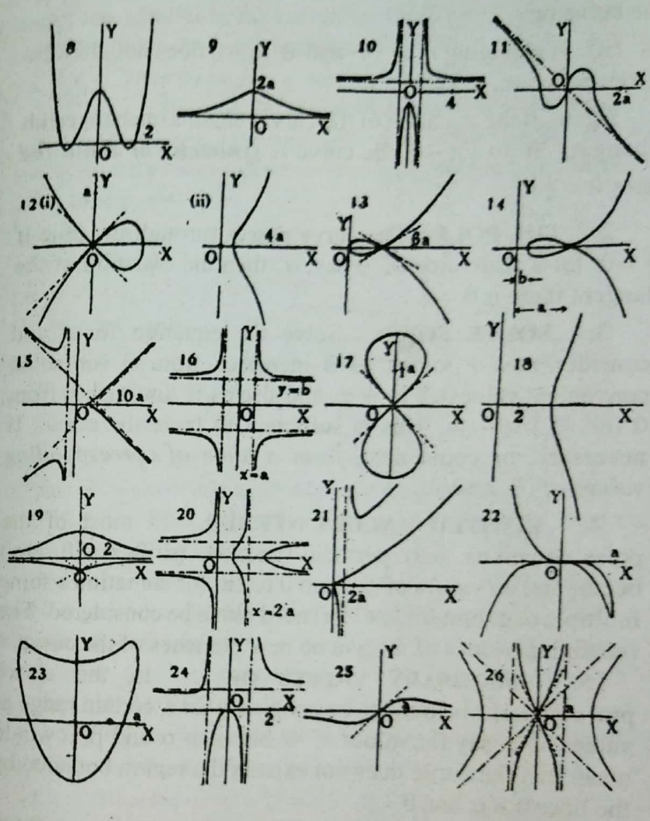
[**Ans.**- Points of inflexion are $(0,0)$, $(a\sqrt{3}, \frac{1}{4}\sqrt{3}a)$, $(-a\sqrt{3}, -\frac{1}{4}\sqrt{3}a)$]

26. Fully examine the nature of the origin on the curve $x^2(x^2-4a^2)=y^2(x^2-a^2)$ and trace the curve.

[**Ans.**- Node at $(0,0)$; tangents at $(0,0)$ are $y=\pm 2x$.]

ANSWERS





11.6. Curve Tracing. Polar Equations.

1. **SYMMETRY.** If changing the sign of θ to $-\theta$ does not alter the equationof the curve, then it is ***symmetrical about the initial line***.

Similarly, the rule for ***symmetry about the line*** $\theta=\frac{1}{2}\pi$ is either of the following :

(i) If changing θ to $\pi-\theta$ does not alter the equation of the curve or

(ii) If changing r to $-r$ and θ to $-\theta$ does not alter the given equation.

Again, if the equation of the curve remains unchanged on changing θ to $\frac{1}{2}\pi-\theta$, the curve is ***symmetrical about the line*** $\theta=\frac{1}{4}\pi$.

2. THE POLE. The curve passes through the pole if $r=0$ for a real value of θ say, α, then the equation of the tangent there is $\theta=\alpha$.

3. SOLVE FOR *r*. Solve the equation for r and consider how r varies as θ increases from 0 (or some convenient value θ_1) to $+\infty$, and also as θ diminishes from 0 (or θ_1) to $-\infty$. This is sufficient to trace the curve. If necessary, or convenient, *form a table of corresponding values of* θ *and* r.

4. LIMITED VALUES NEEDED. In most of the polar equations, only periodic functions ($\sin\theta$, $\cos\theta$, etc.) occur, and so values of θ from 0 to 2π (or sometimes some multiple or submultiple of 2π) need alone be considered. The remaining values of θ give no new branches of the curve.

5. IMAGINARY VALUES OF *r*. In the above procedure, if r is found to be imaginary for a certain range of values of θ, say for values of θ between α and β, it would mean that the curve does not exist in the region bounded by the lines $\theta=\alpha$ and $\theta=\beta$.

6. DIRECTION OF TANGENT. From $\tan\phi=r\,d\theta/dr$ determine ϕ if feasible. This will give the direction of the curve at some representative points.

7. ASYMPTOTE. If the curve possesses an infinite branch, find the asymptote.

If $r\to\infty$ when $\theta\to\alpha$ (either from the left or the right), *it should not be assumed that* $\theta=\alpha$ *is an asymptote.*

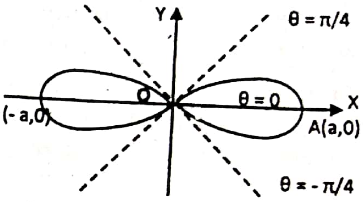
The asymptote might not exist at all; or even if exists, it might be parallel to the radius vector $\theta=\alpha$. The asymptote must be found by an application of § 9.9.

Ex. 1. Trace the curve $r^2=a^2\cos 2\theta$.

[*Agra* **04, 07,** ***Meerut*** **08,** ***Lucknow*** **05, 11,** ***Kumaon*** **13**]

(i) The curve is symmetrical about the initial line and the line $\theta=\frac{1}{2}\pi$ since the equation of the curve remains unchanged on changing θ to $-\theta$ or to $\pi-\theta$.

(ii) Now, $r=0$ when $\cos 2\theta=0$ i.e. $\theta=\pm\frac{1}{4}\pi$.



So, the curve passes through the pole and the tangents to the curve at the pole are $\theta=\pm\frac{1}{4}\pi$, i.e. the pole is a node as the two tangents are not coincident.

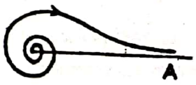
(iii) When $\theta=0$, $r^2=a^2$ or $r=\pm a$ i.e. the curve intersects the initial line at $(a,0)$ and $(-a,0)$.

(iv) Let us consider how r varies as θ varies. As θ increases from 0 to $\frac{1}{4}\pi$, r^2 decreases from a^2 to 0. When θ lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, the value of r^2 is negative i.e. r is imaginary. So the curve does not exist in the region $\frac{1}{4}\pi<\theta<\frac{3}{4}\pi$. Again, when θ increases from $\frac{3}{4}\pi$ to π, r^2 increases from 0 to a^2.

For the region $\pi<\theta<2\pi$, the curve being symmetrical about the initial line, its shape is as shown in the figure.

Ex. 2. Trace the curve $r^2\theta=a^2$.

Consider first the positive value of r. When $\theta=1$, i.e. 1 radian (or about 57°), $r=a$. As θ increases, r continually diminishes. Thus the values of r when $\theta=2\pi, 4\pi, 6\pi, \ldots$ are $0.4\,a$, $0.28\,a$, $0.23\,a, \ldots$. As $\theta\to\infty, r\to 0$. So this part of the curve consists of a spiral in which the curve makes an indefinitely large number of revolutions round the pole, always coming nearer to it.



As θ diminishes from 1, r increases. As $\theta \to 0$ from positive values, $r \to \infty$.

The asymptote, by the method of § 9.9, is $\theta = 0$.

Hence the curve is as shown. This curve is called the Lituus.

To trace the curve for negative values of r, we notice that the point $(-r, \theta)$ is the same as the point $(r, \theta + \pi)$. Hence the branch of the curve which corresponds to $a / r = -|\sqrt{\theta}|$ can be obtained by turning the curve corresponding to the positive values of r through two right angles. For the sake of clearness this branch has not been drawn in the figure.

Ex. 3. Trace the curve $r = a \sin 3\theta,\ a > 0$.

[***Meerut*** **03,** ***Kumaon*** **03,** ***Lucknow*** **04,** ***Rohilkhand*** **12**]

The following table gives corresponding values of θ and r/a.

$3\theta = 0$	Intermediate values	$\frac{1}{2}\pi$	Intermediate values	π	Intermediate values	$\frac{3}{2}\pi$	Intermediate values	2π
$\theta = 0$	"	$\pi/6$	"	$2\pi/6$	"	$3\pi/6$	"	$4\pi/6$
$r/a = 0$	+ve & increasing	1	+ve & decreasing	0	-ve & numerically increasing	-1	-ve & numerically diminishing	0

Now $\tan \phi = r \div dr/d\theta = \frac{1}{3} \tan 3\theta$.

So $\phi = 0$ when $\theta = 0$, i.e. the curve touches the initial line; and $\phi = \frac{1}{2}\pi$ when $\theta = \pi/6$, i.e. the curve is perpendicular to the radius vector there. Also the greatest positive value of r is a. Hence the curve between $\theta = 0$ and $\theta = 120°$ is as shown in Fig. 1. As θ increases further, r/a goes through the same cycle of changes as before, being a maximum at $3\theta = 2\pi + \frac{1}{2}\pi$, i.e., $\theta = 150°$.

Changing the sign of both r and θ does not alter the equation. Hence the curve is symmetrical about the line $\theta = \pi/2$. This also can be used to trace the third loop.

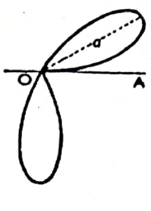


Fig. 1

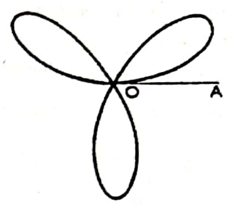


Fig. 2

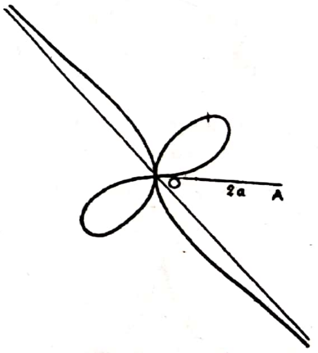
If θ increases beyond π up to 2π, the same branches of the curve are repeated and we do not get any new branch. Hence the complete curve is as shown in Figure 2.

Values of θ outside the range $(0, 2\pi)$ need not be considered, as the value of r/a is periodic.

Ex. 4. Trace the curve $x^5 + y^5 - 5a^2x^2y = 0$.

Transforming to polars we have

$$r^2 = \frac{5a^2 \cos^2 \theta \sin \theta}{\cos^5 \theta + \sin^5 \theta}.$$



Consider first the positive values of r.

When $\theta = 0$, $r = 0$.

r is again zero when $\theta = \frac{1}{2}\pi$. Between these values r is positive.

When θ increases beyond $\frac{1}{2}\pi$, but $\theta < \frac{3}{4}\pi$, r^2 is positive. As θ goes on increasing and thus tends to $\frac{3}{4}\pi$, $r^2 \to \infty$.

For values of θ between $\frac{3}{4}\pi$ and π, r^2 is negative and so r is imaginary.

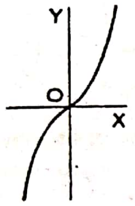
Values of θ between π and 2π need not be considered because of symmetry in opposite quadrants, which is evident from the Cartesian equation which also shows that the tangents at the origin are $x^2y = 0$.

The asymptote can be found to be $x + y = 0$.

Hence the curve is as shown in the figure.

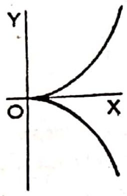
11.7. Some well-known curves. It is desirable that the student should be familiar with some of the well-known curves whose equations and shapes are given below.

Cubical Parabola



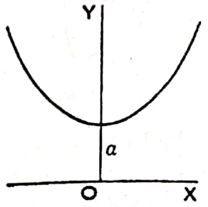
$a^2y = x^3$

Semi-Cubical Parabola



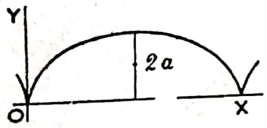
$ay^2 = x^3$

Catenary



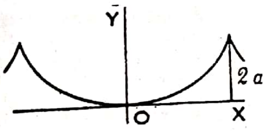
$y = a \cosh (x/a)$

Inverted Cycloid



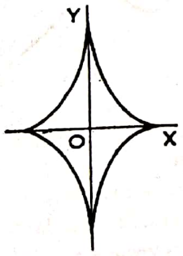
$x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$

Cycloid



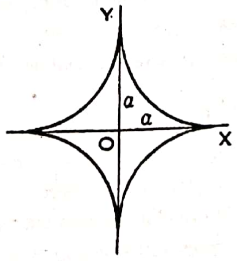
$x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$

Evolute of an Ellipse



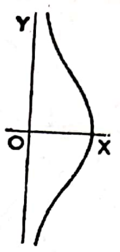
$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

Astroid



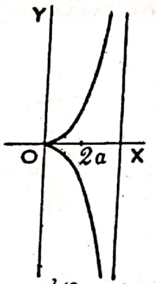
$x^{2/3} + y^{2/3} = a^{2/3}$
or $x = a\cos^3\theta,\ y = a\sin^3\theta$

Witch of Agnesi[1]



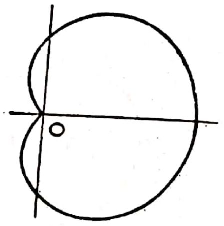
$xy^2 = 4a^2(2a - x)$

Cissoid of Diocles[2]



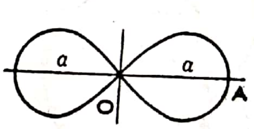
$y^2(2a - x) = x^3$

Cardioid



$r = a(1 + \cos\theta)$

Lemniscate of Bernoulli[3]



$r^2 = a^2 \cos 2\theta$

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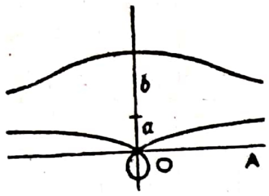
[1] Called after the Italian lady mathematician Maria Gaetna Agnesi (1718-1799) who treated it in her book *Instituzioni Analitiche.* "Agnesi was a somnambulist. Several times, it so happened that she went to her study, while in the somnambulist state, and solved some problem she had left incomplete when awake. In the morning she was surprised to find the solution carefully worked out on paper." - Cajori, *A History of Mathematics.*

[2] Diocles was a Greek mathematician (flourished about 180 B.C.). This curve he used for finding two mean proportionals between two given straight lines.

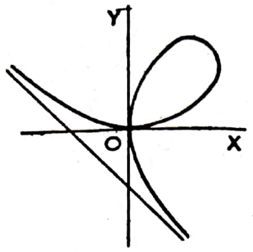
[3] Called after Jakob Bernoulli (1654-1705), professor of mathematics at the University of Basel (Switzerland). He and his brother Johann Bernoulli were staunch friends of Leibnitz and they enriched the calculus immensely in its early days (see Historical Note at the end of this book). The lemniscate was first considered by Jakob Bernoulli in the *Acta eruditorum* (1694). He also considered the logarithmic spiral, which so fascinated him that he willed the curve to be engraved upon his tombstone.

Conchoid of Nicomedes[4]



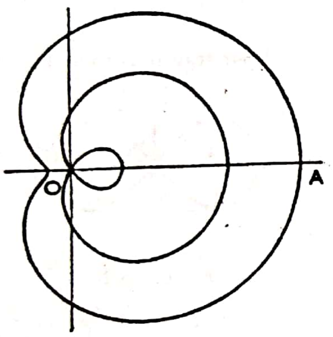
$r = a \operatorname{cosec} \theta \pm b$

Folium of Descartes[5]



$x^3 + y^3 - 3axy = 0$

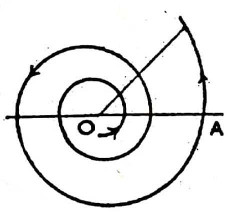
Limacon



$r = a + b\cos\theta$

The two curves shown in the figure correspond to $a < b$ and $a > b$ respectively. The Cardioid is only a particular case of this.

The Logarithmic or the Equiangular Spiral



$r = ae^{m\theta}$

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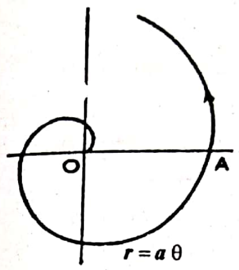
[4] Greek mathematician, Nicomedes and Diocles were contemporaries (about 180 B.C.). Nicomedes devised a little machine by which the conchoid could be easily drawn. He used this curve to trisect angles and construct cubes whose volumes were double the volumes of given cubes.

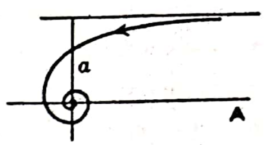
[5] Rene Descartes (1596-1650) was the great French mathematician and philosopher after whom the Cartesian co-ordinates are named. He made many important discoveries. It was in an attack on Fermat's method of finding tangents to curves that he gave the curve $x^3 + y^3 = 3axy$ as a curve to which Fermat's method would not apply. The text was accompanied by a figure which shows that Descartes did not know the correct shape of this curve. The correct shape was first given by C. Huygens (famous Dutch scientist) 54 years later.

Spiral of Archimedes[6]



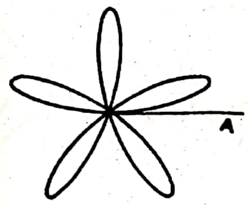
$r = a\theta$

Reciprocal Spiral



$r\theta = a$

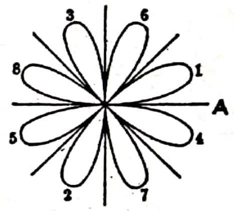
$2m+1$ leaved rose



$r = a\sin(2m+1)\theta$

The case $2m+1=5$ is illustrated in the figure.

$4m$ leaved rose



$r = a\sin 2m\theta$

The case $2m=4$ is illustrated in the figure. Notice that in the curve $r = a\sin n\theta$, if n is odd, there are only n loops, but if n is even, there are $2n$ loops. The order in which the loops are described is indicated in the figure by numbers.

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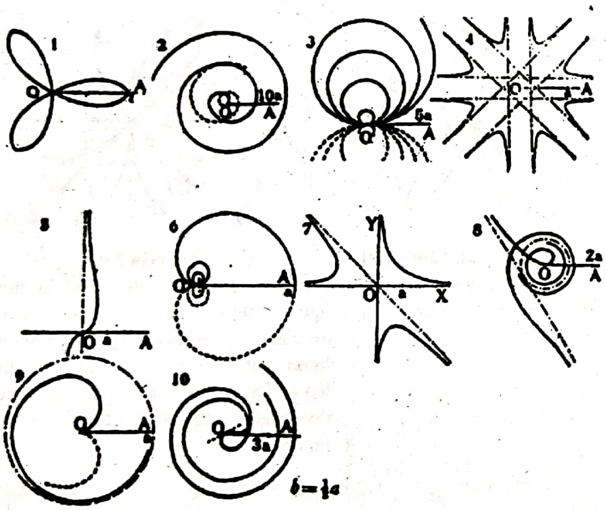
[6] Archimedes (287 - 212 B.C.) of Syracuse (Greece) was the greatest mathematician of antiquity. He wrote on Geometry and Mechanics, and made many fundamental discoveries of the utmost importance. His geometrical proofs were beautifully rigorous. By a special postulate, now known as the postulate of Archimedes, he excluded infinitesimals (see Chap. XV), and his demonstrations were based on the strict theory of limits. However, his postulate did not commend itself to mathematicians until the arithmetical theory of limits was created (see Historical Note).

EXAMPLES

Trace the following curves :

1. $r = a\cos 3\theta$. [*Alld.* 06, *Avadh* 10]
2. $r = a(\theta + \sin\theta)$.
3. $r = a\theta\sin\theta$. [Hint : Trace $r = a\sin\theta$ first.]
4. $r\sin 4\theta = a$.
5. $r^2\cos\theta = a^2\sin^3\theta$.
6. $r\theta = a\sin\theta$.
7. $xy^2 + x^2y = a^3$.
8. $r(1+\theta) = a\theta$.
9. $r(1+\theta^2) = a\theta^2$.
10. $(r-a)^2 = 4ab\theta$.

ANSWERS



12

Partial Differentiation

12.1. Functions of two or more variables. Partial differentiation is a useful tool when there is a need to differentiate functions of two or more variables. We often come across quantities which depend on two or more variables. For example, the volume V of a right circular cylinder of radius r and height h is given by

$$V = \pi r^2 h$$

For a given pair of values of r and h, V has a definite value. Similarly, the area D of a triangle ABC, is given by

$$\Delta = \tfrac{1}{2} ab \sin C,$$

and depends on the three variables a, b, C.

A symbol z which has a definite value for every pair of values of x and y is called a function of the two variables x and y which are called the ***independent variables*** *and z is called the* ***dependent variable.*** A function of x and y is also written as $f(x, y)$.

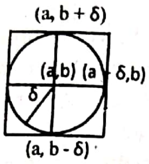
A similar definition can be given for functions of more than two variables.

Strictly speaking, the function denotes the set of all the triples (x, y, z) which can be formed by some given rule $z = f(x, y)$. In most cases x and y can take up any real value (indpendently of each other). The set of all the pairs (x, y) for which z is defined is called the ***domain*** of the function.

The definition given above defines ***single-valued functions.*** Relations like $z^2 = x^2 + y^2$ give two (or more) values of z for every pair of values of x and y. Such functions are called ***multi-valued functions,*** and may be regarded as defining two (or more) functions.

12.11. δ – neighbourhood of a point. If δ be a positive number, then the δ – neighbourhood of a point (a, b) can be defined in two ways :

(1) By the relation $|x-a|<\delta$, $|y-b|<\delta$ which is the square region where x varies from $a-\delta$ to $a+\delta$ but $x \neq a$ and y varies from $b-\delta$ to $b+\delta$ but $y \neq b$.



(2) Alternatively, by the relation $\sqrt{(x-a)^2+(y-b)^2}<\delta$ which is the circular region consisting of a circle of radius δ with its centre at (a, b).

12.12. Limit. A function $f(x, y)$ of two variables is said to tend to the limit L as (x, y) tends to a point (a, b) if, for any arbitrarily chosen $\varepsilon>0$, however small (but not zero), there exists a $\delta>0$ such that for all (x, y) in the δ – neighbourhood of (a, b)

$$|f(x, y)-L|<\varepsilon \text{ whenever } \sqrt{(x-a)^2+(y-b)^2}<\delta$$

or $|f(x, y)-L|<\varepsilon$ whenever $|x-a|<\delta$, $|y-b|<\delta$.

We write this as $\lim\limits_{(x, y) \to (a, b)} f(x, y)=L$.

For a function of one variable, we can approach the point from only two directions (the left and right of the point). But for a function of two variables (say x, y), there are infinitely many paths by which (x, y) can approach the point (a, b).

Thus the limit exists if it is the same along all the paths otherwise not. In other words, the ***limit exists only if it is unique and independent of the path of approach.***

12.13. Continuity. A function $f(x, y)$ of two variables is said to be continuous at a point (a, b) if, for any arbitrarily chosen number $\varepsilon>0$, however small (but not zero), we can find a corresponding number $\delta>0$ such that for all (x, y) in the δ – neighbourhood of (a, b)

$$|f(x, y)-f(a, b)|<\varepsilon \text{ whenever } \sqrt{(x-a)^2+(y-b)^2}<\delta$$

or $|f(x, y)-f(a, b)|<\varepsilon$ whenever $|x-a|<\delta$, $|y-b|<\delta$.

We write this as $\lim\limits_{(x, y) \to (a, b)} f(x, y)=f(a, b)$.

i.e. if the limit of the function exists at (a, b) and is equal to the value of the function there, then $f(x, y)$ is continuous at a point (a, b).

A function is said to be continuous if it is continuous at every point of its domain.

In the subsequent sections, we shall suppose that the functions considered are continuous and their partial differential coefficients exist.

Ex. 1. Evaluate the limit $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x-y}{x+y}$. ***[Lucknow* 2010]**

Let $y=mx$ be the path of approach to the origin, then

$$\lim_{(x, y) \to (0, 0)} \frac{x-y}{x+y}=\lim_{x \to 0} \frac{x-mx}{x+mx}=\lim_{x \to 0} \frac{1-m}{1+m}.$$

We note that the limit depends on the value of m and is different for lines with different slopes, i.e. the limit is not unique and independent of the path of approach. Hence the limit does not exist.

Ex. 2. Examine the continuity at the origin of the function

$$f(x, y)=\frac{xy}{x^2+y^2},\ (x, y) \neq (0, 0);\ f(0, 0)=0.$$

[*Lucknow* 03, *Bilaspur* 07, *Jabalpur* 07, *Avadh* 09, *Raipur* 12, 15]

Let the path of approach to the origin be along the line $y=mx$, then

$$\lim_{(x, y) \to (0, 0)} f(x, y)=\lim_{x \to 0} \frac{x \cdot mx}{x^2+m^2 x^2}=\lim_{x \to 0} \frac{m}{1+m^2}.$$

Thus the limit is not unique since it varies with the value of m, i.e. it is different for different paths.

Hence the limit does not exist and so the function is not continuous at the origin.

Ex. 3. Show that the function $f(x, y)=\dfrac{xy}{\sqrt{x^2+y^2}}$, $(x, y) \neq (0, 0)$; $f(0, 0)=0$, is continuous at the origin.

[*Jabalpur* 06, *Lucknow* 09, 10, 12]

We have $f(x, y)=\dfrac{xy}{\sqrt{x^2+y^2}} < \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$ (1)

since $(x-y)^2 \geq 0$ i.e. $x^2+y^2-2xy \geq 0$

or $x^2+y^2 \geq 2xy > xy$ i.e. $xy < x^2+y^2$ for all $(x, y) \neq (0, 0)$.

Then $|f(x, y)-f(0, 0)| < \sqrt{x^2+y^2}$. [using (1)] (2)

Let us choose $x^2 < \frac{1}{2}\varepsilon^2$ and $y^2 < \frac{1}{2}\varepsilon^2$ i.e. $x < \frac{1}{\sqrt{2}}\varepsilon$ and $y < \frac{1}{\sqrt{2}}\varepsilon$.

Choosing $\delta = \frac{1}{\sqrt{2}}\varepsilon$, we have from (2),

$|f(x, y)-f(0, 0)| < \varepsilon$ whenever $|x-0| < \frac{1}{\sqrt{2}}\varepsilon$ and $|y-0| < \frac{1}{\sqrt{2}}\varepsilon$.

Hence the given function is continuous at the origin.

EXAMPLES

1. Prove by using the definition of limit that

(i) $\lim\limits_{(x, y) \to (1, 2)} (3x+2y)=7$. [*Bilaspur* 06, *Raipur* 11]

(ii) $\lim\limits_{(x, y) \to (1, 1)} (x^2+2y)=3$. [*Rohilkhand* 2009]

(iii) $\lim\limits_{(x, y) \to (0, 0)} \{y\sin(1/x)+x\sin(1/y)\}=0$. [*Himachal* 2007]

Evaluate the following limits :

2. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$.

3. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{xy^3}{x^2+y^6}$. [*Jabalpur* 2005]

4. Show that $\lim\limits_{(x, y) \to (0, 0)} \dfrac{xy^2}{x^2+y^2}=0$ [*Bundelkhand* 2008]

Show that the following limits do not exist:

5. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{2xy}{x^2+y^2}$. [*Lucknow* 2012]

6. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{2x-y}{x^2+y^2}$. [*Bundelkhand* 2012]

7. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$. [*Rohilkhand* 2009]

8. $\lim\limits_{(x, y) \to (0, 1)} \tan^{-1}(y/x)$.

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9. Prove that $f(x, y)=x^2+3y$ is continuous at the point (1, 2). [*Jabalpur* 2004]

Show that the following functions are continuous at the origin :

10. $f(x, y)=\dfrac{x^3-y^3}{x^2+y^2}$, $(x, y) \neq (0, 0)$; $f(0, 0)=0$. [*Lucknow* 2005 (1st part)]

11. $f(x, y)=\dfrac{x^2y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$; $f(0, 0)=0$.

Show that the following functions are not continuous at the origin:

12. $f(x, y)=\dfrac{xy^3}{x^2+y^6}$, $x \neq 0$, $y \neq 0$; $f(0, 0)$. [*Avadh* 06, *Rohilkhand* 07, 10, 11]

13. $f(x, y)=\dfrac{x^2-y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$; $f(0, 0)=0$. [*Bundelkhand* 2010]

14. $f(x, y)=\dfrac{x^4-y^2}{x^4+y^2}$, $(x, y) \neq (0, 0)$; $f(0, 0)=0$.

[HINT: Take $y=mx^2$]

ANSWERS

2. 0. 3. Limit does not exist.

12.2. Partial differential coefficients. The partial differential coefficient of $f(x, y)$ with respect to x is the ordinary differential coefficient of $f(x, y)$ when y is regarded as a constant. It is written as $\dfrac{\partial f}{\partial x}$ or $\partial f / \partial x$.

Thus $$\frac{\partial f}{\partial x}=\lim_{h \to 0} \frac{f(x+h, y)-f(x, y)}{h}$$

provided the limit exists.

Again, the partial differential coefficient $\partial f / \partial y$ of $f(x, y)$ with respect to y is the ordinary differential coefficient of $f(x, y)$ when x is regarded as a constant.

Thus $$\frac{\partial f}{\partial y}=\lim_{k \to 0} \frac{f(x, y+k)-f(x, y)}{k}$$

provided the limit exists.

Similarly, if f is a function of the n variables $x_1, x_2, \ldots\ldots, x_n$, the partial differential coefficient of f with respect to x_1, is the ordinary differential coefficient of f when all the variables except x_1 are regarded as constants, and is written as $\partial f / \partial x_1$.

$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are also denoted by f_x and f_y respectively and are called ***partial differential coefficients of the first order***.

Since f_x and f_y are, generally, functions of x and y, we can again differentiate them partially to get $f_{xx}, f_{xy}, f_{yx}, f_{yy}$;

or $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y\, \partial x}, \dfrac{\partial^2 f}{\partial x\, \partial y}, \dfrac{\partial^2 f}{\partial y^2}$ respectively.

These are called ***partial differential coefficients of the second order***. It should be specially noted that

$$\frac{\partial^2 f}{\partial y\, \partial x} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \text{ and } \frac{\partial^2 f}{\partial x\, \partial y} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

In ordinary cases $\dfrac{\partial^2 f}{\partial y\, \partial x} = \dfrac{\partial^2 f}{\partial x\, \partial y}$.

A proof is given in the next section.

12.21. Commutative property of partial differential coefficients. We shall prove that a sufficient conditon for the equality of $\dfrac{\partial^2 f}{\partial x\, \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ is that these partial differential coefficients be ***continuous***.

Consider the expression

$$\frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk}. \quad (1)$$

Let $\quad F(x) = f(x, y+k) - f(x, y) \quad (2)$

Then the numerator of (1) is $F(x+h) - F(x)$. Keeping y constant and applying Lagrange's mean value theorem, we see that

$$F(x+h) - F(x) = h F_x(x + \theta_1 h), \quad 0 < \theta_1 < 1,$$

or $\quad F(x+h) - F(x) = h\{f_x(x+\theta_1 h, y+k) - f_x(x+\theta_1 h, y)\}$

by (2). Thus (1) becomes

$$\{f_x(x+\theta_1 h, y+k) - f_x(x+\theta_1 h, y)\}/k. \quad (3)$$

Regarding (3) as a function of y, and applying the mean value theorem, the expression (3) becomes

$$k\, f_{xy}(x+\theta_1 h, y+\theta_2 k)/k, \quad 0<\theta_2<1$$

where f_{xy} means $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$.

So (1) becomes $\quad f_{xy}(x+\theta_1 h, y+\theta_2 k) \quad (4)$

But if we first let $\varphi(y)$ stand for $f(x+h, y) - f(x, y)$ and thus apply to (1) the mean value theorem by regarding it first as a function of y (x being kept constant) and then as a function of x, we can show that (1) is the same as

$$f_{yx}(x+\theta_3 h, y+\theta_4 k) \quad (5)$$

Thus the expressions (4) and (5) are equal. Taking limits as $h \to 0$ and $k \to 0$, and keeping in mind that the functions (4) and (5) are continuous, we get

$$f_{xy}(x, y) = f_{yx}(x, y) \text{ which was to be proved.}$$

With the help of the proposition: $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$

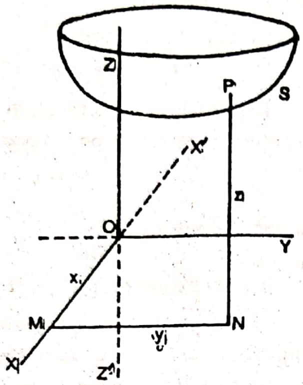
it is easy to see that

$$\frac{\partial^3 u}{\partial x^2\, \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}u = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial x}u = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}u = \frac{\partial^3 u}{\partial y\, \partial x^2}.$$

Similarly $\quad \dfrac{\partial^{m+n} u}{\partial x^m\, \partial y^n} = \dfrac{\partial^{m+n} u}{\partial y^n\, \partial x^m}$

and the proposition can be readily extended if there are more than two variables.

12.22. Geometrical representation of a function of two variables. To indicate the position of a point in space we most often use rectangular Cartesian co- ordinates. Let $X'OX$ and $Y'OY$ be two perpendicular lines intersecting at O and let $Z'OZ$ be a third line through O perpendicular to the plane containing the first two.



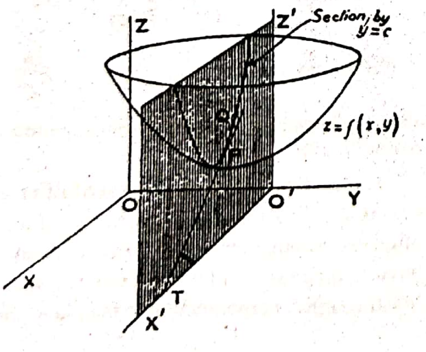
Then the three lines OX, OY, OZ can be taken as the axes of coordinates.

Let P be any point in space. Draw from P the perpendicular PN to the plane XOY. From the foot N of the perpendicular, draw the line NM parallel to OY cutting OX in M. Let x, y, z be the lengths of OM, MN, NP respectively. Then x, y, z are the rectangular Cartesian coordinates of P, and are generally written as (x, y, z).

Let S be a given surface. Take arbitrary values of x and y and plot the point $N(x, y, 0)$. From N draw a vertical line NP cutting the surface S in P, and let $NP = z$. Then z depends on x and y and we may write
$$z = f(x, y). \qquad (1)$$

This is known as the equation of the surface S. Conversely, any equation of type (1) represents a surface. The equation could also be in the implicit form $F(x, y, z) = 0$.

12.23. Geometrical Meaning of partial differential coefficients. Consider a point $P(x, y, z)$ in space, then the equation $z = f(x, y)$ represents a surface.



The equation $y = c$, where c is a constant, will represent the plane parallel to the plane XOZ and at distance c from it.

This is because all points for which the value of y is evidently lie on this plane. Hence if we consider only those points of the surface $z = f(x, y)$ for which $y = c$, we shall get the curve in which the plane $y = c$ cuts the surface $z = f(x, y)$.

Let the plane $y = c$ cut OY in O', and take $O'X'$ to be parallel to OX. If P be a given point on the section of the surface $z = f(x, y)$ by the plane $y = c$ and Q any other point on it, it is clear from the figure that as $Q \to P$, QP tends to the tangent at P to the section, and the tangent of its inclination to $O'X'$ tends to
$$\lim_{h \to 0} \frac{f(x+h, c) - f(x, c)}{h} \quad \text{i.e. to} \quad \left(\frac{\partial f}{\partial x}\right)_{y=c}$$

Hence $\partial z / \partial x$ at P is equal to the tangent of the angle which the tangent at P to the section of the surface $z = f(x, y)$, by a plane through P parallel to the plane XOZ, makes with a line drawn parallel to the axis of x.

A similar interpretation can be given to $\partial z / \partial y$.

Ex. 1. Find the partial differential coefficients of $x^4 + x^2y^2 + y^4$.

Let $\quad f(x, y) = x^4 + x^2y^2 + y^4$, then

$f_x = 4x^3 + 2xy^2, \quad f_y = 2x^2y + 4y^3, \quad f_{xy} = 4xy,$

$f_{yx} = 4xy, \quad f_{xx} = 12x^2 + 2y^2, \quad f_{yy} = 2x^2 + 12y^2,$ etc.

Notice that in the above $f_{xy} = f_{yx}$.

Ex. 2. If $u = 1/\sqrt{x^2 + y^2 + z^2}$, $x^2 + y^2 + z^2 \neq 0$

show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

[***Bundelkhand* 05, *Patna* 06, *Magadh* 09, *Kanpur* 12**]

Now $\quad \dfrac{\partial u}{\partial x} = -\dfrac{1}{2}(x^2 + y^2 + z^2)^{-3/2} . 2x = -x(x^2 + y^2 + z^2)^{-3/2}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = -(x^2 + y^2 + z^2)^{-3/2} - x(-\tfrac{3}{2})(x^2 + y^2 + z^2)^{-5/2} . 2x$$
$$= -(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2} \qquad (1)$$

Similarly $\quad \dfrac{\partial^2 u}{\partial y^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3y^2(x^2 + y^2 + z^2)^{-5/2} \qquad (2)$

and $$\frac{\partial^2 u}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3z^2 (x^2+y^2+z^2)^{-5/2} \qquad (3)$$

Adding (1), (2) and (3) we get the result.

Ex. 3. If $x^x y^y z^z = c$, then show that $\frac{\partial^2 z}{\partial x\, \partial y} = -(x \log ex)^{-1}$, when $x = y = z$. **[*Delhi* 03, *Indore* 04, *Kashi Vidyapeeth* 09, *Garhwal* 09, *Avadh* 06, 08, 10, *Raipur* 13, *Bundelkhand* 14]**

We have $x^x y^y z^z = c$

Taking the logarithm of both sides, we get

$$x \log x + y \log y + z \log z = \log c \qquad (1)$$

Differentiating (1) partially w.r.t. y,

$$y \cdot \frac{1}{y} + 1 \cdot \log y + \left(z \cdot \frac{1}{z} + 1 \cdot \log z\right) \frac{\partial z}{\partial y} = 0$$

i.e. $$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z} \qquad (2)$$

Similarly, $$\frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}$$

Again, differentiating (2) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{-(1+\log x)(-1)}{(1+\log z)^2} \cdot \frac{1}{z} \frac{\partial z}{\partial y} = \frac{(1+\log x)}{(1+\log z)^2} \cdot \frac{1}{z} \cdot \frac{-(1+\log y)}{1+\log z}$$

$$= -\frac{(1+\log x)^2}{(1+\log x)^3} \cdot \frac{1}{x} \qquad [\text{since } x = y = z]$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x \log ex}$$

which is the required result.

Ex. 4. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

(i) $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}.$$

[*Garhwal* 08, *Jabalpur* 08, *Raipur* 11]

(ii) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

[*Kanpur* 07, *Garhwal* 08, *Avadh* 09, *Agra* 09, *Kumaon* 04, 10, *Kashi Vidyapeeth* 08, 10, *Bundelkhand* 10, *Rohilkhand* 07, 10, 11]

We have $u = \log(x^3 + y^3 + z^3 - 3xyz)$, so that.

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz},$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz},$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}.$$

Adding these three equations, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} = \frac{3}{x+y+z}.$$

(ii) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ using the result of (i) above}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}.$$

Ex. 5. If $u = f(r)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

[*Delhi* 06, *Rohilkhand* 08, *Indore* 08, *Lucknow* 02, 07, 10]

Differentiating $r^2 = x^2 + y^2$ partially w.r.t. x, we have

$$2r \frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}. \qquad (1)$$

Again differentiating $u = f(r)$ partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{du}{dr} \frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r} \qquad [\text{using (1)}] \qquad (2)$$

Differentiating (2) again partially w.r.t. x gives

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(f'(r)\cdot\frac{x}{r}\right)=\frac{f'(r)}{r}+x\frac{\partial}{\partial x}\left(\frac{f'(r)}{r}\right)$$

$$=\frac{f'(r)}{r}+x\left(\frac{f''(r)}{r}-\frac{f'(r)}{r^2}\right)\frac{\partial r}{\partial x}$$

$x/z = 1/z$

$$=\frac{f'(r)}{r}+\frac{x^2}{r^2}\left(f''(r)-\frac{f'(r)}{r}\right) \qquad \text{[using (1)]}$$

$x/y = 1/y$

or $$\frac{\partial^2 u}{\partial x^2}=\frac{1}{r^2}\left\{r\,f'(r)+x^2 f''(r)-x^2\frac{f'(r)}{r}\right\}. \qquad (3)$$

Similarly $$\frac{\partial^2 u}{\partial y^2}=\frac{1}{r^2}\left\{r\,f'(r)+y^2 f''(r)-y^2\frac{f'(r)}{r}\right\}. \qquad (4)$$

Adding (3) and (4), we obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{r^2}\left\{2r\,f'(r)+(x^2+y^2)f''(r)-\frac{x^2+y^2}{r}f'(r)\right\}$$

$$=\frac{1}{r^2}\{r^2 f''(r)+2r\,f'(r)-r\,f'(r)\}, \text{ since } x^2+y^2=r^2$$

$\partial y/\partial x$

$$=f''(r)+\frac{1}{r}f'(r).$$

EXAMPLES

1. Illustrate the theorem $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$ when u is equal to

~~(i)~~ $ax^2+2hxy+by^2$. ~~(ii)~~ $x\sin y+y\sin x$.

~~(iii)~~ $x\log y$. ~~(iv)~~ $\log\{(x^2+y^2)/xy\}$.

(v) $\log\tan(y/x)$. (vi) $(ay-bx)/(by-ax)$.

2. If $f(x,y)=x^2\tan^{-1}(y/x)-y^2\tan^{-1}(x/y), x\neq 0, y\neq 0$, $f(0,0)=0$, then prove that at the origin $\dfrac{\partial^2 f}{\partial x\,\partial y}\neq\dfrac{\partial^2 f}{\partial y\,\partial x}$.

3. Let $f(x,y)=\dfrac{x\,y\,(x-y)}{x+y}, (x,y)\neq(0,0); f(0,0)=0$, show that $f_{xy}(0,0)\neq f_{yx}(0,0)$ **[*Himachal* 2006]**

4. If $u=e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}=(1+3xyz+x^2y^2z^2)e^{xyz}$.

[*Gorakhpur* 03, *Jabalpur* 05, *Indore* 07, *Lucknow* 10, *Avadh* 03, 11, *Kanpur* 11, *Kashi Vidyapeeth* 12, *Raipur* 12]

5. If $u=\tan^{-1}\dfrac{xy}{\sqrt{1+x^2+y^2}}$, show that $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{1}{(1+x^2+y^2)^{3/2}}$. **[*Raipur* 03, *Indore* 05]**

6. If $z=f(x+ay)+\phi(x-ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2}=a^2\dfrac{\partial^2 z}{\partial x^2}$.

[*Kanpur* 05, *Indore* 05, *Lucknow* 11]

7. If $z=x\,f(x+y)+y\,\phi(x+y)$, prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=2\frac{\partial^2 z}{\partial x\,\partial y}.$$

8. If $z(x+y)=x^2+y^2$, show that

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right).$$

[*Delhi* 07, *Bundelkhand* 06, 11, 12, *Raipur* 13]

9. If $u=\sin(\sqrt{x}+\sqrt{y})$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}(\sqrt{x}+\sqrt{y})\cos(\sqrt{x}+\sqrt{y}). \qquad \textbf{[Agra 2011]}$$

10. If $u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

$\dfrac{1}{\sqrt{1-x^2}}$ $\dfrac{1}{1+x^2}$

[*Bilaspur* 06, *Magadh* 07, *Lucknow* 09, *Kumaon* 12]

11. If $u=x^2\tan^{-1}\dfrac{y}{x}-y^2\tan^{-1}\dfrac{x}{y}$, prove that $\dfrac{\partial^2 u}{\partial y\,\partial x}=\dfrac{x^2-y^2}{x^2+y^2}$.

[*Lucknow* 03, *Bilaspur* 07, *Raipur* 13]

12. If $V=(x^2+y^2+z^2)^{-1/2}$, prove that

(i) $x\dfrac{\partial V}{\partial x}+y\dfrac{\partial V}{\partial y}+z\dfrac{\partial V}{\partial z}=-V$. **[*Jabalpur* 03, *Bundelkhand* 05]**

(ii) $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=0$. **[*Bundelkhand* 05, *Kumaon* 08]**

13. If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$$ [*Delhi* 03, 08]

14. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}.$$ [*Avadh* 09, *Patna* 09]

15. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z).$$

[*Garhwal* 04, *Purvanchal* 05]

12.3. Homogeneous Functions.

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n$$

is an expression in which every term is of degree n. It is called a ***homogeneous function of degree n***. This can also be written as

$$x^n\left\{a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_n\left(\frac{y}{x}\right)^n\right\}$$

Hence the general definition of a homogeneous function is as follows :

$x^n f(y/x)$ is called a homogeneous function of degree n, whatever the function f may be.

Thus $x^2 \sin(y/x)$ is a homogeneous function of x and y of degree 2.

Generally, if the function $f(x_1, x_2, \ldots, x_m)$ of the m variables $x_1, x_2, \ldots, x_m$ can be expressed in the form

$$x_r^n\, F\left(\frac{x_1}{x_r}, \frac{x_2}{x_r}, \ldots, \frac{x_m}{x_r}\right),$$

then $f(x_1, x_2, \ldots, x_m)$ is called a homogeneous function of $x_1, x_2, \ldots, x_m$ of degree n.

12.31. Euler's Theorem on Homogeneous Functions.

If $f(x, y)$ be a homogeneous function of x and y of degree n, then

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.}$$

[*Himachal* 05, *Bundelkhand* 06, *Garhwal* 06, *Bilaspur* 06, *Avadh* 08,09, *Magadh* 04,06,10, *Patna* 10, *Kashi Vidyapeeth* 10]

Let $f(x, y)$ be expressed in the form $x^n F(y/x)$, since it is a homogeneous function of degree n. Then, we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{x^n F(y/x)\} = nx^{n-1}F(y/x) + x^n F'(y/x).\frac{-y}{x^2}$$

or $$x\frac{\partial f}{\partial x} = nx^n F(y/x) - x^{n-1}y.F'(y/x) \qquad (1)$$

Again $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{x^n F(y/x)\} = x^n F'(y/x).\frac{1}{x}$$

or $$y\frac{\partial f}{\partial y} = x^{n-1}y\,F'(y/x) \qquad (2)$$

Adding (1) and (2), we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\,x^n F(y/x)\} = n\,f(x, y).$$

In general, if $f(x_1, x_2, \ldots, x_m)$ be a homogeneous function of degree n, then

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_m\frac{\partial f}{\partial x_m} = nf.$$

The proof is similar to that for two variables.

12.32. Corollary. If u be a a homogeneous function of x and y of degree n, then

(i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x}$. [*Lucknow* 2009]

(ii) $x\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$.

(iii) $x\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)\,u$. [*Patna* 08, 11]

Since u is a a homogeneous function of degree n, then by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu. \tag{1}$$

(i) Differentiating both sides of (1) partially with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}=n\frac{\partial u}{\partial x},$$

i.e.,

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=(n-1)\frac{\partial u}{\partial x}. \tag{2}$$

x

(ii) Differentiating both sides of (1) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y},$$

i.e.

$$x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}. \tag{3}$$

y

(iii) Multiplying (2) by x and (3) by y, and then adding the resulting equations, we have

$$x\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=(n-1)\,n\,u=n(n-1)\,u, \text{ using (1).}$$

Ex. 1. Verify Euler's theorem when

$$f(x,y,z)=3x^2yz+5xy^2z+4z^4.$$

Here $\dfrac{\partial f}{\partial x}=6xyz+5y^2z.$

Therefore $x\dfrac{\partial f}{\partial x}=6x^2yz+5xy^2z$

Similarly $y\dfrac{\partial f}{\partial y}=3x^2yz+10xy^2z.$

and $z\dfrac{\partial f}{\partial z}=3x^2yz+5xy^2z+16z^4.$

Hence $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z}=12x^2yz+20xy^2z+16z^4$

$$=4f(x,y,z),$$

which verifies Euler's theorem for a function of degree 4 in this case.

Ex. 2. If $u=\log\left\{\dfrac{x^2+y^2}{x+y}\right\}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1.$

[*Kanpur* 06, *Delhi* 07, *Lucknow* 08]

Here $u=\log\left\{\dfrac{x^2+y^2}{x+y}\right\}=\log z$

where $z=e^u=\dfrac{x^2+y^2}{x+y}=x\dfrac{1+(y/x)^2}{1+(y/x)}$

i.e. z is a homogeneous function of x, y of degree 1.

Hence by Euler's theorem, we have $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z.$ (1)

Then $\dfrac{\partial z}{\partial x}=e^u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y}=e^u\dfrac{\partial u}{\partial y}$ (2)

or $e^u x\dfrac{\partial u}{\partial x}+e^u y\dfrac{\partial u}{\partial y}=e^u$ [from (1) and (2)]

or $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1.$

Ex. 3. If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, $x\neq y$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$$

[*Magadh* 06, *Lucknow* 10, *Patna* 10, *Avadh* 11, *Kanpur* 04, 12]

and $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=(1-4\sin^2 u)\sin 2u.$

[*Delhi* 03, *Kumaon* 03, *Rohilkhand* 03, *Himachal* 06]

Here $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}=\tan^{-1}z$

where $z=\tan u=\dfrac{x^3+y^3}{x-y}=x^2\dfrac{1+(y/x)^3}{1-(y/x)}$ (1)

which is a homogeneous function of x, y of degree 2.

Hence by Euler's theorem, we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$ (2)

Since $\frac{\partial z}{\partial x}=\sec^2 u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$ [from (1)] (3)

$\therefore$ $x\sec^2 u\frac{\partial u}{\partial y}+y\sec^2\frac{\partial u}{\partial y}=2\tan u$ [from (2) & (3)]

or $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u=\sin 2u$ (4)

Differentiating (4) partially w.r.t. x and w.r.t. to y, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}=2\cos 2u\,.\frac{\partial u}{\partial x}$$

and $$x\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=2\cos 2u\,.\frac{\partial u}{\partial y}$$

Multiplying the first equation by x, the second by y, and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\cdot\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=(1-4\sin^2 u)\sin 2u.$$ [using (4)]

Ex. 4. If $u=x\phi(y/x)+\psi(y/x)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

[*Garhwal* 05, *Bundelkhand* 05, *Himachal* 07, *Kanpur* 08, *Kumaon* 11]

Let $u_1=x\phi(y/x)$ and $u_2=\psi(y/x)$ so that $u=u_1+u_2$.

Here u_1 is a homogeneous function of degree 1 and u_2 is of degree zero, so that by Euler's theorem,

$$x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}=u_1 \text{ and } x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}=0. \quad (1)$$

Now $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\frac{\partial}{\partial x}(u_1+u_2)+y\frac{\partial}{\partial y}(u_1+u_2)$$

$$=\left(y\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}\right)+\left(x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}\right)=u_1.$$ [from (1)]

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u_1. \quad (2)$$

Differentiating (2) partially w.r.t. first x and then y, we have

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial u_1}{\partial x} \quad (3)$$

and $$x\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=\frac{\partial u_1}{\partial y} \quad (4)$$

Multiplying (3) by x and (4) by y, adding and using (1) and (2), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+u_1=u_1.$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

EXAMPLES

1. Verify Euler's theorem in the following cases:
   (i) $f(x, y)=ax^2+2hxy+by^2$.
   (ii) $f(x, y, z)=ayz+bxz+cxy$. **[*Lucknow* 11, 12]**
2. Verify Euler's Theorem for the function
   (i) $u=(x^{1/4}+y^{1/4})/(x^{1/5}+y^{1/5})$. **[*Jabalpur* 2006]**
   (ii) $z=\sin u$ where $u=\sin^{-1}(\sqrt{x^2+y^2})/(x+y)$.
3. If $u=\log\frac{x^4+y^4}{x+y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$. **[*Bilaspur* 09]**
4. If $u=x^3+y^3+z^3+3xyz$, show that
   $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3u.$$
5. If $u=\cos^{-1}(x+y)/(\sqrt{x}+\sqrt{y})$, prove that
   $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$ **[*Delhi* 05, *Rohilkhand* 10]**
6. If $u=\cos^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\cot u$.
   [*Gorakhpur* 2003]

7. If $u = \sin^{-1} \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$.

[***Delhi*** **04,** ***Indore*** **05,** ***Gorakhpur*** **05,** ***Rohilkhand*** **07, 11,** ***Lucknow*** **11]**

8. If $u = \sin^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{1}{2} \tan u$.

[***Magadh*** **03,** ***Lucknow*** **05,** ***Purvanchal*** **05,** ***Bilaspur*** **07]**

9. If $u = \sin^{-1} \dfrac{x^2 + y^2}{x + y}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \tan u$.

[***Gorakhpur*** **03,** ***Patna*** **03, 05,** ***Magadh*** **05,** ***Meerut*** **05,** ***Bilaspur*** **08,** ***Avadh*** **06, 10, 11,** ***Raipur*** **15]**

10. If $z = \sec^{-1} \dfrac{x^3 + y^3}{x + y}$, show that $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = 2 \cot z$.

11. If $u = \sin^{-1} \{(x^{1/3} + y^{1/3}) / (x^{1/2} + y^{1/2})\}$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \tfrac{1}{144} \tan u \, (13 + \tan^2 u).$$

[***Indore*** **04,** ***Agra*** **09,** ***Garhwal*** **10]**

12. If $u = \tan^{-1} \dfrac{x^2 + y^2}{x + y}$, prove that

(i) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \tfrac{1}{2} \sin u$. [***Kumaon*** **03,** ***Kashi Vidyapeeth*** **11]**

(ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \, \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = (1 - 4 \sin^2 u) \sin 2u$.

[***Indore*** **2005]**

13. If $u = \dfrac{x^2 y^2}{x^2 + y^2}$, show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u.$$

14. If $V = \log_e \sin \left\{ \dfrac{\pi (2x^2 + y^2 + xz)^{1/2}}{2 (x^2 + xy + 2yz + z^2)^{1/3}} \right\}$, find the value of

$x \dfrac{\partial V}{\partial x} + y \dfrac{\partial V}{\partial y} + z \dfrac{\partial V}{\partial z}$ at $x = 0, y = 1, z = 2$ [***Kanpur*** **2010]**

[Ans. $\frac{1}{12} \pi$]

12.4. Total Differential Coefficient. If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$, then we can find the value of u in terms of t by substituting from the last two equations in the first equation. Here u is a ***composite function*** of the single variable t.

Then du/dt is called the ***total differential coefficient*** of u, to distinguish it from the partial differential coefficients $\partial u / \partial x$ and $\partial u / \partial y$.

To find du/dt without actually substituting the values of x and y in $f(x, y)$, we can proceed as follows:

Let δt be a small increment in t corresponding to which the increments in x, y and u may be taken as δx, δy and δu.

Then $u + \delta u = f(x + \delta x, y + \delta y)$ so that

$$\frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y)}{\delta t}$$

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}$$

on adding and subtracting $f(x, y + \delta y)$

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Taking limits as $\delta t \to 0$ and hence δx and δy also tend to zero, we have

$$\frac{du}{dt} = \lim_{t \to 0} \frac{\delta u}{\delta t} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \lim_{\delta t \to 0} \frac{\delta x}{\delta t} + \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \lim_{\delta t \to 0} \frac{\delta y}{\delta t}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}. \quad (1)$$

where $$\lim_{\delta t\to 0}\frac{\delta x}{\delta t}=\frac{dx}{dt} \text{ and } \lim_{\delta t\to 0}\frac{\delta y}{\delta t}=\frac{dy}{dt}.$$

Also, if δy does not depend on δx, then, by definition

$$\lim_{\delta x\to 0}\frac{f(x+\delta x, y+\delta y)-f(x, y+\delta y)}{\delta x}=\frac{\partial f\,(x, y+\delta y)}{\partial x} \qquad (2)$$

Moreover, supposing that $\partial f(x,y)/\partial x$ is a continuous function of y.

$$\lim_{\delta y\to 0}\frac{\partial f(x, y+k)}{\partial x}=\frac{\partial f\,(x,y)}{\partial x}=\frac{\partial f}{\partial x}. \qquad (3)$$

Therefore, from (2) and (3), we have

$$\lim_{\delta x\to 0}\frac{f(x+\delta x, y+\delta y)-f(x, y+\delta y)}{\delta x}=\frac{\partial f}{\partial x}.$$

Hence, from (1) $$\frac{du}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt};$$

i.e. $$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

In general, if $u=f(x_1, x_2, \ldots\ldots, x_n)$ where $x_1, x_2, \ldots\ldots, x_n$ are all functions of t, we can prove that

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt}+\ldots+\frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt}.$$

This is sometimes also called the ***chain rule of partial differentiation.***

12.41. Change of Variables. If $u=f(x, y)$ where

$$x=\phi(t_1, t_2) \text{ and } y=\psi(t_1, t_2) \qquad (1)$$

it is frequently necessary to change expressions involving u, x, y, into expressions involving u, t_1, t_2 where x and y are now functions of the two variables t_1 and t_2 instead of only one variable t. Hence, by § 12.4 above, we get

$$\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1} \text{ and } \frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}.$$

In case equations (1) are easily solvable for t_1 and t_2 in terms of x and y, say

$$t_1=F_1(x, y) \text{ and } t_2=F_2(x, y),$$

it is easier to use the formulae

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x}+\frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x},\\ \frac{\partial u}{\partial y}&=\frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y},\end{aligned}\right\}$$

The above formulae can be easily extended to the case of more than two independent variables.

Ex. Transform the equation

$$\frac{\partial^2 z}{\partial x^2}+2xy^2\frac{\partial z}{\partial x}+2(y-y^3)\frac{\partial z}{\partial y}+x^2y^2z=0$$

by the substitution $x=uv,\ y=1/v$. [*Kumaon* **2012**]

Since z is a function of x, y which in turn are functions of u, v, therefore z is a function of u, v.

From the given relations $x=uv,\ y=1/v$, we have

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\cdot 0=\frac{1}{v}\frac{\partial z}{\partial u}$$

So $$\frac{\partial^2 z}{\partial x^2}=\frac{1}{v}\frac{\partial}{\partial u}\left(\frac{1}{v}\frac{\partial z}{\partial u}\right)=\frac{1}{v^2}\frac{\partial^2 z}{\partial u^2}.$$

Also $$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}-\frac{1}{y^2}\frac{\partial z}{\partial v}=v\left(u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}\right).$$

Substituting the above values in the given equation, we get

$$\frac{1}{v^2}\frac{\partial^2 z}{\partial u^2}+2\frac{u}{v}\left(\frac{1}{v}\frac{\partial z}{\partial u}\right)+2v\left(\frac{1}{v}-\frac{1}{v^3}\right)\left(u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}\right)+u^2z=0$$

or $$\frac{1}{v^2}\frac{\partial^2 z}{\partial u^2}+2\frac{u}{v^2}\frac{\partial z}{\partial u}+\frac{2u}{v^2}(v^2-1)\frac{\partial z}{\partial u}-\frac{2}{v}(v^2-1)\frac{\partial z}{\partial v}+u^2z=0$$

or $$\frac{\partial^2 z}{\partial u^2}+[2u+2u(v^2-1)]\frac{\partial z}{\partial u}-2v(v^2-1)\frac{\partial z}{\partial v}+u^2v^2z=0$$

on multiplying throughout by v^2

or $$\frac{\partial^2 z}{\partial u^2}+2uv^2\frac{\partial z}{\partial u}+2(v-v^3)\frac{\partial z}{\partial v}+u^2v^2z=0.$$

The student should note that this equation is exactly the same as the given equation except for the variables. This shows that the function z is the same for u and v as it is for x and y.

12.5. An important case. Differentiation of implicit functions. When $f(x, y)$ is a function of x and y, and y *is a function of* x, the total (i.e. the ordinary) differential coefficient of f with respect to x is given by

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}.$$

Now, if y be an implicit function of x of the form

$$f(x, y)=c$$

where c is a constant, the above formula becomes

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0 \text{ which gives}$$

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}.$$

This is an important formula for finding the differential coefficient of an implicit function.

Again, if f is a function of the n variables $x_1, x_2, x_3, \ldots\ldots, x_n$, and $x_2, x_3, \ldots\ldots x_n$, are all functions of x_1, the total (i.e. the ordinary) differential coefficient of f with respect to x_1 is given by

$$\frac{df}{dx_1}=\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dx_1}+\frac{\partial f}{\partial x_3}\cdot\frac{dx_3}{dx_1}+\ldots+\frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dx_1}$$

12.51. The second differential coefficient of an implicit function. If $f(x, y)=0$, we have seen that

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} \qquad (1)$$

We can find d^2y/dx^2 by differentiating this.

For the sake of convenience $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\,\partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ are often denoted by p, q, r, s and t respectively.

Adopting this notation, we have, from (1), $\frac{dy}{dx}=-\frac{p}{q}$.

Differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=-\frac{q\frac{dp}{dx}-p\frac{dq}{dx}}{q^2} \qquad (2)$$

But by § 12.5, $$\frac{dp}{dx}=\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}\cdot\frac{dy}{dx}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\,\partial y}\cdot\frac{dy}{dx}$$

$$=r+s\left(-\frac{p}{q}\right)=\frac{qr-ps}{q}.$$

Similarly $$\frac{dq}{dx}=\frac{\partial q}{\partial x}+\frac{\partial q}{\partial y}\cdot\frac{dy}{dx}=s+t\left(-\frac{p}{q}\right)=\frac{qs-pt}{q}$$

Substituting these values in (2), we get

$$\frac{d^2y}{dx^2}=-\frac{q^2r-2qps+p^2t}{q^3}.$$

Ex. 1. If $u=x^4y^5$, where $x=t^2$ and $y=t^3$, find du/dt. Verify by substitution.

We have $$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$=4x^3y^5.2t+5x^4y^4.3t^2.$$

As a verification we notice that this value of du/dt

$$=4(t^2)^3(t^3)^5.2t+5(t^2)^4(t^3)^4.3t^2=23t^{22}.$$

Also, substituting the values of x and y in u, we have

$$u=(t^2)^4(t^3)^5=t^{23}.$$

Therefore $du/dt=23t^{22}$, as before.

Ex. 2. If $x^3+3x^2y+6xy^2+y^3=1$, find dy/dx.

Let $f(x,y)=x^3+3x^2y+6xy^2+y^3-1$

Then $\partial f/\partial x = 3x^2+6xy+6y^2$, $\partial f/\partial y = 3x^2+12xy+3y^2$

Hence $$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}=-\frac{x^2+2xy+2y^2}{x^2+4xy+y^2}.$$

Ex. 3. If $u=x^2-y^2+\sin yz$ where $y=e^x$ and $z=\log x$, find du/dx.

We have $$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}+\frac{\partial u}{\partial z}.\frac{dz}{dx}$$

$$=2x+(-2y+z\cos yz)e^x+(y\cos yz)/x$$

It is easy to verify that we shall get the same value of du/dx if we first convert u into a function of x alone by substituting in it the values of y and z, and then differentiate.

Ex. 4. Find d^2y/dx^2 if $ax^2+2hxy+by^2=1$.

Here $p=2(ax+hy)$, $q=2(hx+by)$,

$r=2a$, $s=2h$, $t=2b$.

Hence

$$\frac{d^2y}{dx^2}=-\frac{(hx+by)^2a-2(ax+hy)(hx+by)h+(ax+hy)^2b}{(hx+by)^3}.$$

Ex. 5. If $u=f(x-y,y-z,z-x)$, prove that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$.

[***Bundelkhand* 06, *Lucknow* 05, 08, *Kanpur* 09, *Rohilkhand* 09, *Agra* 07, 10, *Kashi Vidyapeeth* 10, *Garhwal* 10, *Avadh* 10**]

Let $X=x-y$, $Y=y-z$, $Z=z-x$, then we have,

$$\frac{\partial X}{\partial x}=1,\quad \frac{\partial X}{\partial y}=-1,\quad \frac{\partial X}{\partial z}=0$$

$$\frac{\partial Y}{\partial x}=0,\quad \frac{\partial Y}{\partial y}=1,\quad \frac{\partial Y}{\partial z}=-1$$

$$\frac{\partial Z}{\partial x}=-1,\quad \frac{\partial Z}{\partial y}=0,\quad \frac{\partial Z}{\partial z}=1$$

Since $u=f(X,Y,Z)$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}$$

$$=\frac{\partial u}{\partial X}(1)+\frac{\partial u}{\partial Y}(0)+\frac{\partial u}{\partial Z}(-1)=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z} \qquad (1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}$$

$$=\frac{\partial u}{\partial X}(-1)+\frac{\partial u}{\partial Y}(1)+\frac{\partial u}{\partial Z}(0)=-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y} \qquad (2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}$$

$$=\frac{\partial u}{\partial X}(0)+\frac{\partial u}{\partial Y}(-1)+\frac{\partial u}{\partial Z}(1)=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z} \qquad (3)$$

Adding (1), (2) and (3), we get $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$.

EXAMPLES

1. Find dy/dx if (i) $ax^2+2hxy+by^2=1$.

   (ii) $y^x+x^y=c$. [***Agra* 2008**]

2. If $u=x\log xy$, where $x^3+y^3+3xy=1$, find du/dx.

3. Find du/dx if $u=\sin(x^2+y^2)$, where $a^2x+b^2y^2=c^2$.

4. If $y^3-3ax^2+x^3=0$, then prove that $\dfrac{d^2y}{dx^2}+\dfrac{2a^2x^2}{y^5}=0$.

5. If $f(x,y)=0$, $\phi(y,z)=0$, show that

   $$\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}.$$ [***Avadh* 2006**]

6. If the curves $f(x,y)=0$, $\phi(y,z)=0$ touch, show that at the point of contact $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y}-\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \phi}{\partial x}=0$ [***Lucknow* 07, 09**]

   [HINT: At the point of contact, the value of dy/dx for the common tangent will be the same for both curves.]

7. If $z=f(u,v)$, $u=x^2-2xy-y^2$, $v=y$, show that

   $(x+y)\dfrac{\partial z}{\partial x}+(x-y)\dfrac{\partial z}{\partial y}=0$ is transformed into $\partial z/\partial v=0$.

8. If z is a function of x and y, and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, then prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$. **[*Lucknow* 2012]**

9. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}$,

where $x = \xi \cos \alpha - \eta \sin \alpha$, $y = \xi \sin \alpha + \eta \cos \alpha$. **[*Kumaon* 2009]**

10. If $u = e^x \sec y$, $v = e^x \tan y$ and V is a function of u, v, show that

$$\cos y\left(\frac{\partial^2 V}{\partial x\, \partial y} - \frac{\partial V}{\partial y}\right) = uv\left(\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right) + (u^2 + v^2)\frac{\partial^2 V}{\partial u\, \partial v}.$$

11. If $x + y = 2e^\theta \cos \phi$ and $x - y = 2ie^\theta \sin \phi$, where $i = \sqrt{-1}$,

show that $\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial \phi^2} = 4xy\frac{\partial^2 v}{\partial x\, \partial y}$. **[*Lucknow* 05, 09]**

ANSWERS

1. (i) $-(ax + hy)/(hx + by)$.
   (ii) $-(y^x \log y + yx^{y-1})/(xy^{x-1} + x^y \log x)$.
2. $1 + \log xy - x(x^2 + y)/y(x + y^2)$.
3. $2x\{\cos(x^2 + y^2\}(1 - a^2/b^2)$.

12.6. Change of variables from Cartesian to polar coordinates. A change from Cartesian coordinates (x, y) to polars (r, θ) is very often required and vice versa.

We have $x = r\cos\theta$, $y = r\sin\theta$, then

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}(y/x).$$

It must be remembered that in this connection that just as $\partial/\partial x$ always means $(\partial/\partial x)_{y = \text{const.}}$ and $\partial/\partial y$ means $(\partial/\partial y)_{x = \text{const.}}$, similarly $\partial/\partial r$ means $(\partial/\partial r)_{\theta = \text{const.}}$ and $\partial/\partial\theta$ means $(\partial/\partial\theta)_{r = \text{const.}}$.

Thus $x = r\cos\theta$, so $\partial x/\partial r = \cos\theta$.

But, although $r = x\sec\theta$, $\partial r/\partial x \neq \sec\theta$,

because this value of $\partial r/\partial x$ is really $(\partial r/\partial x)_{\theta = \text{const.}}$ and we want $(\partial r/\partial x)_{y = \text{const}}$.

In fact, since $r^2 = x^2 + y^2$, we have

$$2r\frac{\partial r}{\partial x} = 2x,\quad \text{i.e. } \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta.,$$

The simple rule is to express x and y in polars before finding the partial differential coefficients of x or y, and to express r and θ in Cartesian coordinates before finding the partial differential coefficients of r or θ.

Ex. Transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar coordinates. **[*Garhwal* 2004]**

OR If V is a function of two variables x and y and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$ **[*Avadh* 07, 09]**

Since $x = r\cos\theta$, $y = r\sin\theta$, (1)

Then $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$. (2)

Hence $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\theta$, $\frac{\partial r}{\partial y} = \frac{r}{\sqrt{x^2 + y^2}} = \sin\theta$.

[using (1) & (2)]

$$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin\theta}{r},\quad \frac{\partial\theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\cos\theta}{r}.$$

[using (1) & (2)]

$\therefore$ By § 12.41, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$

or $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)$,

i.e. $\frac{\partial u}{\partial x} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)u.$

It follows that $\frac{\partial^2 u}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}.$$

Similarly $\frac{\partial^2 u}{\partial y^2}=\sin^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$

$$+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}.$$

Adding, we get $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$

Hence the transformed equation is $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0.$

EXAMPLES

1. If $x=r\cos\theta,\ y=r\sin\theta$, prove that

(i) $\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$. (ii) $\frac{1}{r}\frac{\partial x}{\partial\theta}=r\frac{\partial\theta}{\partial x}$. (iii) $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0.$

(iii) $\frac{\partial^2\theta}{\partial x\,\partial y}=-\frac{\cos 2\theta}{r^2}$. **[*Lucknow* 2012]**

2. If $x=r\cos\theta,\ y=r\sin\theta,\ z=f(x,y)$, then prove that

$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2.$ **[*Garhwal* 2005]**

3. If $x=r\cos\theta,\ y=r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial y\,\partial x}\right)^2$ **[*Avadh* 2008]**

(ii) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\}.$ **[*Lucknow* 07, 11]**

4. If $x=r\cos\theta,\ y=r\sin\theta,\ z=f(x,y)$, prove that

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial z}{\partial\theta}\sin\theta.$ **[*Avadh* 2006]**

5. If $u=t^n e^{-r^2/4t}$, find the value of n for which

$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}.$ **[Ans. $-\frac{3}{2}$]** **[*Lucknow* 2011]**

12.7. Taylor's Theorem for functions of two variables.

By Taylor's theorem for functions of a single variable,

$$f(x+h,y+k)=f(x,y+k)+h\frac{\partial f(x,y+k)}{\partial x}$$

$$+\frac{h^2}{2!}\frac{\partial^2 f(x,y+k)}{\partial x^2}+\dots.$$

Expanding now each term,

$$f(x+h,y+k)=f(x,y)+k\frac{\partial f(x,y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2}+\dots.$$

$$+h\frac{\partial}{\partial x}\left\{f(x,y)+k\frac{\partial f(x,y)}{\partial y}+\dots\right\}$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x,y)+k\frac{\partial f(x,y)}{\partial y}+\dots\right\}+\dots.$$

Hence $f(x+h,y+k)=f(x,y)+h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}$

$$+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)+\dots.$$

This is sometimes written symbolically as

$$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f+\dots.$$

$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f+\dots.$$

Generally, if f is a function of the n variables $x_1,x_2,\dots,x_n$, we shall have similarly

$$f(x_1+h_1,x_2+h_2,\dots\dots,x_n+h_n)=f(x_1,x_2,\dots,x_n)$$

$$+\left(h_1\frac{\partial f}{\partial x_1}+h_2\frac{\partial f}{\partial x_2}+\dots.+h_n\frac{\partial f}{\partial x_n}\right)+\dots.$$

In the neighbourhood of a point (a,b), Taylor's theorem can also be expressed as follows :

$$f(x,y)=f(a,b)+\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a,b)$$

$$+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)+\ldots.$$

NOTE. $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f$ means that we expand $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f$ as if ∂, ∂x and ∂y were algebraic quantities and write f on the right-hand side of each term, but after this $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, etc., are no longer considered as the products or quotients of powers of ∂ and ∂x or ∂y multiplied by f. Thus $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$ means $h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}$.

But all the laws of algebra cannot be applied in the symbolical method of writing. For, $\frac{\partial}{\partial x}f\neq f\frac{\partial}{\partial x}$, although $\frac{a}{bc}f=f\frac{a}{bc}$.

The consideration of the remainder after terms of degree n in h and k, and of the conditions under which the infinite expansion is valid, is beyond the scope of this book.

Ex. Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$.

[*Jabalpur* 03, *Rohilkhand* 09, *Avadh* 07, 10]

Let $f(x,y)=x^2y+3y-2$, then $f(1,-2)=-10$ at the point $(1,-2)$.

The partial derivatives of $f(x,y)$ are

$$\frac{\partial f}{\partial x}=2xy,\quad \frac{\partial f}{\partial y}=x^2+3,\quad \frac{\partial^2 f}{\partial x^2}=2y,\quad \frac{\partial^2 f}{\partial y^2}=0,\quad \frac{\partial^2 f}{\partial x\,\partial y}=2.$$

Partial derivatives of $f(x,y)$ of the third and higher orders are zero.

At the point $(1,-2)$, we have

$$\frac{\partial f}{\partial x}=-4,\quad \frac{\partial f}{\partial y}=4,\quad \frac{\partial^2 f}{\partial x^2}=-4,\quad \frac{\partial^2 f}{\partial y^2}=0,\quad \frac{\partial^2 f}{\partial x\,\partial y}=2.$$

Thus by Taylor's theorem about the point $(1,-2)$, we get

$$f(x,y)=f(1,-2)+\left[(x-1)\frac{\partial}{\partial x}+(y+2)\frac{\partial}{\partial y}\right]f(1,-2)$$

$$+\frac{1}{2!}\left[(x-1)\frac{\partial}{\partial x}+(y+2)\frac{\partial}{\partial y}\right]^2 f(1,-2)+0$$

since terms of higher orders are zero.

$$=f(1,-2)+\left[(x-1)\frac{\partial}{\partial x}+(y+2)\frac{\partial}{\partial y}\right]f(1,-2)$$

$$+\frac{1}{2!}\left[(x-1)^2\frac{\partial^2}{\partial x^2}+2(x-1)(y+2)\frac{\partial^2}{\partial x\,\partial y}+(y+2)^2\frac{\partial^2}{\partial y^2}\right]^2 f(1,-2)$$

or $\quad x^2y+3y-2=-10+[(x-1)(-4)+(y+2)\cdot 4]$

$$+\tfrac{1}{2}[(x-1)^2(-4)+2(x-1)(y+2)\cdot 2+(y+2)^2\cdot 0]$$

or $\quad x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)$.

EXAMPLES

1. Expand the function $f(x,y)=x^2+xy+y^2$ in powers of $(x-2)$ and $(y-3)$. **[*Jiwaji* 03, *Jabalpur* 04, 05, *Indore* 08]**

2. Expand the function $f(x,y)=x^2+xy-y^2$ by Taylor's theorem in powers of $(x-1)$ and $(y+2)$. **[*Bilaspur* 07, *Jabalpur* 08]**

Expand by Taylor's theorem upto the second degree terms:

3. $f(x,y)=\log(x+e^y)$ in powers of $(x-1)$ and y.

4. $\sin xy$ in powers of $(x-1)$ and $(y-\frac{1}{2}\pi)$. **[*Avadh* 2009]**

5. Expand $f(x,y)=e^x\cos y$ in powers of x and y such that it includes all terms upto the third degree.

ANSWERS

1. $x^2+xy+y^2=19+7(x-2)+8(y-3)$
$\qquad +(x-2)^2+(x-2)(y-3)+(y-3)^2$

2. $x^2+xy-y^2=-5+5(y+2)+(x-1)^2+(x-1)(y+2)-(y+2)^2$.

3. $\log(x+e^y)=\log 2+\frac{1}{2}[(x-1)+y]-\frac{1}{8}[(x-1)^2+2(x-1)y-y^2]$.

4. $\sin xy=1-\frac{1}{8}\pi^2(x-1)^2-\frac{1}{2}\pi(x-1)(y-\frac{1}{2}\pi)-\frac{1}{2}(y-\frac{1}{2}\pi)^2$.

5. $e^x\cos y=1+x+\frac{1}{2}(x^2-y^2)+\frac{1}{6}(x^3-3xy^2)$.

13

Jacobians

13.1. Jacobians. If u and v are functions of two independent variables x and y, the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is called the ***Jacobian*** of u, v with respect to x, y ; and is written as $\dfrac{\partial(u,v)}{\partial(x,y)}$ or sometimes as $J(u, v)$.

In general, if $u_1, u_2, u_3, \ldots, u_n$ are functions of n independent variables $x_1, x_2, x_3, \ldots, x_n$, the Jacobian of $u_1, u_2, u_3, \ldots, u_n$ with respect to $x_1, x_2, x_3, \ldots, x_n$ is

$$J(u_1, u_2, \ldots u_n) = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\[2ex] \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\[2ex] \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} & \cdots & \dfrac{\partial u_3}{\partial x_n} \\[2ex] \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \dfrac{\partial u_n}{\partial x_3} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

An application of Jacobians will be found useful in the change of variables of multiple integrals.

13.11. A Special case. If the functions $u_1, u_2, \ldots, u_n$ of $x_1, x_2, \ldots, x_n$, are of the form

$$u_1 = f_1(x_1);\ u_2 = f_2(x_1, x_2), \ldots,\ u_n = f_n(x_1, x_2, x_3, \ldots, x_n),$$

then it is evident that

$$\frac{\partial u_1}{\partial x_2} = 0,\ \frac{\partial u_1}{\partial x_3} = 0, \ldots,\ \frac{\partial u_1}{\partial x_n} = 0,$$

$$\frac{\partial u_2}{\partial x_3} = 0,\ \frac{\partial u_2}{\partial x_4} = 0,\ \ldots,\ \frac{\partial u_2}{\partial x_n} = 0 \quad \text{and so on.}$$

Then, substitution of the above values in the Jacobian gives

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\[2ex] \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & 0 & \cdots & 0 \\[2ex] \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

Thus it can be stated that ***if the functions $u_1, u_2, \ldots, u_n$ of $x_1, x_2, \ldots, x_n$, are of the form $u_1 = f_1(x_1)$, $u_2 = f_2(x_1, x_2), \ldots,$ $u_n = f_n(x_1, x_2, \ldots, x_n)$, then the Jacobian***

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdots\cdots \frac{\partial u_n}{\partial x_n}.$$

Ex. 1. If $u_1 = x_2 x_3 / x_1$, $u_2 = x_3 x_1 / x_2$, $u_3 = x_1 x_2 / x_3$, show that $J(u_1, u_2, u_3) = 4$.

[*Meerut* 04, *Lucknow* 04, *Indore* 05, *Rohilkhand* 07, *Kashi Vidyapeeth* 08, *Agra* 08, *Garhwal* 09, *Raipur* 14, *Bundelkhand* 14]

We have $\qquad J(u_1, u_2, u_3) = \dfrac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$

$$=\begin{vmatrix}\dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3}\\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3}\\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3}\end{vmatrix}=\begin{vmatrix}-\dfrac{x_2x_3}{x_1^{\,2}} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1}\\ \dfrac{x_3}{x_2} & -\dfrac{x_3x_1}{x_2^{\,2}} & \dfrac{x_1}{x_2}\\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1x_2}{x_3^{\,2}}\end{vmatrix}$$

$$=\frac{1}{x_1^{\,2}}\cdot\frac{1}{x_2^{\,2}}\cdot\frac{1}{x_3^{\,2}}\begin{vmatrix}-x_2x_3 & x_1x_3 & x_1x_2\\ x_2x_3 & -x_1x_3 & x_1x_2\\ x_2x_3 & x_1x_3 & -x_1x_2\end{vmatrix}$$

$$=\frac{x_2x_3\cdot x_1x_3\cdot x_1x_2}{x_1^{\,2}x_2^{\,2}x_3^{\,2}}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=\begin{vmatrix}0 & 0 & 2\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix},\text{ by } R_1\to R_1+R_2$$

$$=2\,(1+1)=4.$$

Ex. 2. If $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta,$ show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta.$$

[***Meerut*** **03,** ***Kumaon*** **04,** ***Purvanchal*** **05,** ***Garhwal*** **07,** ***Avadh*** **07,** ***Rohilkhand*** **09, 12,** ***Bundelkhand*** **13]**

We have $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta,$

$$\therefore\quad \frac{\partial x}{\partial r}=\sin\theta\cos\phi,\quad \frac{\partial x}{\partial\theta}=r\cos\theta\cos\phi,\quad \frac{\partial x}{\partial\phi}=-r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r}=\sin\theta\sin\phi,\quad \frac{\partial y}{\partial\theta}=r\cos\theta\sin\phi,\quad \frac{\partial x}{\partial\phi}=r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r}=\cos\theta,\quad \frac{\partial z}{\partial\theta}=-r\sin\theta,\quad \frac{\partial z}{\partial\phi}=0$$

$$\therefore\quad \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} & \dfrac{\partial x}{\partial\phi}\\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta} & \dfrac{\partial y}{\partial\phi}\\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial\theta} & \dfrac{\partial z}{\partial\phi}\end{vmatrix}$$

$$=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$=\sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi)-\sin\theta\sin\phi\,(-r^2\sin^2\theta\sin\phi)$$
$$+\cos\theta\,(r^2\sin\theta\cos\theta\cos^2\phi+r^2\sin\theta\cos\theta\sin^2\phi)$$

$$=(r^2\sin^3\theta\cos^2\phi+r^2\sin^3\theta\sin^2\phi)$$
$$+r^2\sin\theta\cos^2\theta\,(\cos^2\phi+\sin^2\phi)$$

$$=\ r^2\sin^3\theta+r^2\sin\theta\cos^2\theta$$

$$=r^2\sin\theta\,(\sin^2\theta+\cos^2\theta)=\ r^2\sin\theta\,.$$

EXAMPLES

1. If $u=1-x,\ v=x\,(1-y),\ w=xy\,(1-z),$ prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=-x^2y$$

[*Kanpur* 2007]

2. If $x=r\cos\theta,\ y=r\sin\theta,$ find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$ and $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

[*Meerut* 04, *Agra* 06, *Kanpur* 12]

3. If $x=u\,(1+v),\ y=v\,(1+u)$, then find the value of $\dfrac{\partial(x,y)}{\partial(u,v)}$.

[*Lucknow* 2011]

4. If $x=u\,(1-v),\ y=v\,(1-u)$, find the Jacobian of x,y with respect to u,v. **[*Kanpur* 2006]**

5. If $u=xyz,\ v=xy+yz+zx,\ w=x+y+z,$ show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=(x-y)(y-z)(z-x)$$

[*Lucknow* 2010]

6. If $y_1=\cos x_1,\ y_2=\sin x_1\cos x_2,\ y_3=\sin x_1\sin x_2\cos x_3,$

then show that $\dfrac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}=-\sin^3x_1\sin^2x_2\sin x_3$.

[*Jabalpur* 04, *Indore* 06]

7. Find the Jacobian $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$, given that

$x=r\cos\theta\cos\phi,\ y=r\sin\theta\sqrt{1-m^2\sin^2\phi},\ z=r\sin\phi\sqrt{1-n^2\sin^2\theta},$ where $m^2+n^2=1$.

ANSWERS

2. r and $1/r$. **3.** $1+u+v$. **4.** $1-u-v$.

7. $\dfrac{r^2(1-m^2\sin^2\phi-n^2\sin^2\theta)}{\sqrt{(1-m^2\sin^2\phi)(1-n^2\sin^2\theta)}}$.

13.2. Jacobian of Functions of Functions. *If $u_1, u_2, \ldots, u_n$ are functions of $y_1, y_2, \ldots, y_n$ and $y_1, y_2, \ldots, y_n$ are functions of $x_1, x_2, \ldots, x_n$, then*

$$\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}=\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(y_1,y_2,\ldots,y_n)}\cdot\frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)}.$$

Since $u_1, u_2, \ldots, u_n$ are functions of $y_1, y_2, \ldots, y_n$ which are again functions of $x_1, x_2, \ldots, x_n$, therefore

$$\frac{\partial u_1}{\partial x_1}=\frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1}+\frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1}+\ldots\frac{\partial u_1}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_1}=\sum_{r=1}^{n}\frac{\partial u_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_1}.$$

$$\frac{\partial u_1}{\partial x_2}=\frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2}+\frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2}+\ldots\frac{\partial u_1}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_2}=\sum_{r=1}^{n}\frac{\partial u_1}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_2}.$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\qquad\text{and so on till}\qquad(1)$$

$$\frac{\partial u_n}{\partial x_n}=\frac{\partial u_n}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_n}+\frac{\partial u_n}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_n}+\ldots\frac{\partial u_n}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_n}=\sum_{r=1}^{n}\frac{\partial u_n}{\partial y_r}\cdot\frac{\partial y_r}{\partial x_n}.$$

Then $\dfrac{\partial(u_1,u_2,\ldots,u_n)}{\partial(y_1,y_2,\ldots,y_n)}\times\dfrac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)}$

$$=\begin{vmatrix}\frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \ldots & \frac{\partial u_1}{\partial y_n}\\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \ldots & \frac{\partial u_2}{\partial y_n}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial y_2} & \ldots & \frac{\partial u_n}{\partial y_n}\end{vmatrix}\times\begin{vmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \ldots & \frac{\partial y_1}{\partial x_n}\\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \ldots & \frac{\partial y_2}{\partial x_n}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \ldots & \frac{\partial y_n}{\partial x_n}\end{vmatrix}$$

$$=\begin{vmatrix}\sum\frac{\partial u_1}{\partial y_r}\frac{\partial y_r}{\partial x_1} & \sum\frac{\partial u_1}{\partial y_r}\frac{\partial y_r}{\partial x_2} & \ldots & \sum\frac{\partial u_1}{\partial y_r}\frac{\partial y_r}{\partial x_n}\\ \sum\frac{\partial u_2}{\partial y_r}\frac{\partial y_r}{\partial x_1} & \sum\frac{\partial u_2}{\partial y_r}\frac{\partial y_r}{\partial x_2} & \ldots & \sum\frac{\partial u_2}{\partial y_r}\frac{\partial y_r}{\partial x_n}\\ \ldots & \ldots & \ldots & \ldots\\ \sum\frac{\partial u_n}{\partial y_r}\frac{\partial y_r}{\partial x_1} & \sum\frac{\partial u_n}{\partial y_r}\frac{\partial y_r}{\partial x_2} & \ldots & \sum\frac{\partial u_n}{\partial y_r}\frac{\partial y_r}{\partial x_n}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \ldots & \frac{\partial u_1}{\partial x_n}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \ldots & \frac{\partial u_2}{\partial x_n}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \ldots & \frac{\partial u_n}{\partial x_n}\end{vmatrix}\qquad\text{[using (1)]}$$

$$=\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}.$$

NOTE. The above formula is very similar to the formula

$$\frac{df}{dx}=\frac{df}{dt}\times\frac{dt}{dx}$$

for the function of a function. It will be found that several other formulae for Jacobians show similarities with formulae for derivatives of a single variable.

13.21. A Special Case. Taking $u_1=x_1;\, u_2=x_2, \ldots, u_n=x_n$, we have

$$\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(x_1,x_2,\ldots,x_n)}=\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(y_1,y_2,\ldots,y_n)}\cdot\frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)},$$

so that $\dfrac{\partial(x_1,x_2,\ldots,x_n)}{\partial(y_1,y_2,\ldots,y_n)}\cdot\dfrac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)}=1.$

13.3. Jacobian of Implicit Functions. *If $u_1, u_2, \ldots, u_n$ and $x_1, x_2, \ldots, x_n$ are connected implicitly by n equations given by*

$$f_1(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0,$$

$$f_2(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0,$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$f_n(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0, \textit{ then}$$

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(u_1, u_2, \ldots, u_n)} \cdot \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

Differentiating the given equations with respect to $x_1, x_2, \ldots, x_n$, we have

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \ldots + \frac{\partial f_1}{\partial u_n}\frac{\partial u_n}{\partial x_1} = 0,$$

$$\frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial f_1}{\partial u_n}\frac{\partial u_n}{\partial x_2} = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{ and so on till}$$

$$\frac{\partial f_n}{\partial x_n} + \frac{\partial f_n}{\partial u_1}\frac{\partial u_1}{\partial x_n} + \frac{\partial f_n}{\partial u_2}\frac{\partial u_2}{\partial x_n} + \ldots + \frac{\partial f_n}{\partial u_n}\frac{\partial u_n}{\partial x_n} = 0$$

or
$$\sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} = -\frac{\partial f_1}{\partial x_1}, \quad \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} = -\frac{\partial f_1}{\partial x_2}, \ldots$$

$$\ldots \quad \ldots \text{ and so on till } \sum \frac{\partial f_n}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_n} = -\frac{\partial f_n}{\partial x_n}. \qquad (1)$$

Then
$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(u_1, u_2, \ldots, u_n)} \cdot \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \ldots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \ldots & \frac{\partial f_2}{\partial u_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \ldots & \frac{\partial f_n}{\partial u_n} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \ldots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \ldots & \frac{\partial u_2}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \sum \frac{\partial f_1}{\partial u_r}\frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_1}{\partial u_r}\frac{\partial u_r}{\partial x_2} & \ldots & \sum \frac{\partial f_1}{\partial u_r}\frac{\partial u_r}{\partial x_n} \\ \sum \frac{\partial f_2}{\partial u_r}\frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_2}{\partial u_r}\frac{\partial u_r}{\partial x_2} & \ldots & \sum \frac{\partial f_2}{\partial u_r}\frac{\partial u_r}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \sum \frac{\partial f_n}{\partial u_r}\frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_n}{\partial u_r}\frac{\partial u_r}{\partial x_2} & \ldots & \sum \frac{\partial f_n}{\partial u_r}\frac{\partial u_r}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} & \ldots & -\frac{\partial f_1}{\partial x_n} \\ -\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} & \ldots & -\frac{\partial f_2}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ -\frac{\partial f_n}{\partial x_1} & -\frac{\partial f_n}{\partial x_2} & \ldots & -\frac{\partial f_n}{\partial x_n} \end{vmatrix} \qquad \text{[using (1)]}$$

$$= (-1)^n \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} = (-1)^n \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

The above result can also be written as

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} \Bigg/ \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(u_1, u_2, \ldots, u_n)}. \qquad (2)$$

13.31. A Special Case. If the implicit relationship between $u_1, u_2, \ldots, u_n$ and $x_1, x_2, \ldots, x_n$ is of the type

$$f_1(x_1, x_2, \ldots, x_n, u_1) = 0$$

$$f_1(x_2, x_3, \ldots, x_n, u_1, u_2) = 0$$

$$f_1(x_3, x_4, \ldots, x_n, u_1, u_2, u_3) = 0$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$f_1(x_n, u_1, u_2, \ldots, u_n) = 0$$

then $$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(u_1, u_2, \ldots, u_n)} = \frac{\partial f_1}{\partial u_1} \cdot \frac{\partial f_2}{\partial u_2} \cdots \frac{\partial f_n}{\partial u_n} \qquad \text{[using § 13.11]}$$

and $$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}.$$

Hence from (2), we have

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} \Big/ \frac{\partial f_1}{\partial u_1} \cdot \frac{\partial f_2}{\partial u_2} \cdots \frac{\partial f_n}{\partial u_n}.$$

Ex. 1. If $u = (x+y)/(1-xy)$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\partial(u, v)/\partial(x, y)$. [*Lucknow* 03, *Meerut* 05]

Also find the relationship between u and v.

[*Himachal* 07, *Rohilkhand* 11]

Let $x = \tan \phi$ and $y = \tan \theta$ so that $\phi = \tan^{-1} x$, $\theta = \tan^{-1} y$. (1)

Then $$u = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \cdot \tan \theta} = \tan(\phi + \theta) \text{ and } v = \phi + \theta. \qquad (2)$$

Hence $$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(\phi, \theta)} \times \frac{\partial(\phi, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \theta} \end{vmatrix} \times \begin{vmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \sec^2(\phi+\theta) & \sec^2(\phi+\theta) \\ 1 & 1 \end{vmatrix} \times \begin{vmatrix} \frac{1}{1+x^2} & 0 \\ 0 & \frac{1}{1+y^2} \end{vmatrix} \qquad \text{[from (1) \& (2)]}$$

$$= 0 \times \left(\frac{1}{1+x^2} \cdot \frac{1}{1+y^2} - 0 \right) = 0.$$

From (2), the relationship between u and v is given by

$$u = \tan(\phi + \theta) = \tan v.$$

Ex. 2. Prove that $$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1.$$

[*Avadh* 06, *Kanpur* 08, *Bundelkhand* 08, 09, 10, 11, *Raipur* 11]

Differentiating partially w.r.t. x and y, we have

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for $u = u(x, y)$, $$1 = \frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u}, \quad 0 = \frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial v} \qquad (1)$$

and for $v = v(x, y)$, $$0 = \frac{\partial v}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial u}, \quad 1 = \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial v} \qquad (2)$$

$$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u} & \frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial u} & \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

[using (1) and (2)]

Ex. 3. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, prove that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2uv(u - v)}.$$

[*Sagar* 03, *Garhwal* 03, *Avadh* 08, *Rohilkhand* 08, *Agra* 11, *Bundelkhand* 06, 12, *Lucknow* 12]

Let $f_1 \equiv u^3 + v^3 - (x + y)$ and $f_2 \equiv u^2 + v^2 - (x^3 + y^3)$.

Then $$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)} \Big/ \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}. = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} \div \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= \frac{3y^2 - 3x^2}{6uv(u - v)} = \frac{y^2 - x^2}{2uv(u - v)}.$$

Ex. 4. If $x + y + z = u$, $y + z = uv$ and $z = uvw$, find the Jacobian of x, y, z with respect to u, v, w.

[*Kumaon* 05, *Himachal* 06, *Lucknow* 04, 08, *Rohilkhand* 10]

OR

If $x+y+z=u,\ y+z=uv$ and $z=uvw$, prove that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v.$

[***Bundelkhand*** **05, 06,** ***Bilaspur*** **06,** ***Lucknow*** **06,** ***Agra*** **08,** ***Rohilkhand*** **05, 09,** ***Raipur*** **15]**

Let $f_1 \equiv x+y+z-u,\ f_2 \equiv y+z-uv$ and $f_3 \equiv z-uvw.$

Since it is a special case of implicit functions, the Jacobian is given by

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=(-1)^3\left(\frac{\partial f_1}{\partial u}\cdot\frac{\partial f_2}{\partial v}\cdot\frac{\partial f_3}{\partial w}\right)\Big/\left(\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial y}\cdot\frac{\partial f_3}{\partial z}\right)$$

$$=-\frac{(-1)(-u)(-uv)}{1\times 1\times 1}=u^2v.$$

Ex. 5. The roots of equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ in λ are $u,v,w.$ Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=-2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}.$$

[***Lucknow*** **07,** ***Kumaon*** **08,** ***Garhwal*** **08,** ***Kanpur*** **10]**

Rearranging the given equation in terms of λ, we have

$$3\lambda^3-3\lambda^2(x+y+z)+3\lambda(x^2+y^2+z^2)-(x^3+y^3+z^3)=0$$

Since the roots of this equation are u,v,w, by Algebra, the sum and products of the roots are given by

$$u+v+w=x+y+z,\qquad uv+vw+wu=x^2+y^2+z^2$$

and $\quad uvw=\frac{1}{3}(x^3+y^3+z^3).$

Let $\quad f_1 \equiv u+v+w-x-y-z,$

$\quad f_2 \equiv uv+vw+wu-x^2-y^2-z^2$

and $\quad f_3 \equiv uvw-\frac{1}{3}(x^3+y^3+z^3).$

Then $\quad \dfrac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\Big/\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$$=-\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z}\end{vmatrix}\div\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w}\\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w}\end{vmatrix}$$

$$=\begin{vmatrix}-1 & -1 & -1\\ -2x & -2y & -2z\\ -x^2 & -y^2 & -z^2\end{vmatrix}+\begin{vmatrix}1 & 1 & 1\\ v+w & w+u & u+v\\ vw & uw & uv\end{vmatrix}$$

$$=\begin{vmatrix}-1 & 0 & 0\\ -2x & 2(x-y) & 2(x-z)\\ -x^2 & x^2-y^2 & x^2-z^2\end{vmatrix}\div\begin{vmatrix}1 & 0 & 0\\ v+w & u-v & u-w\\ vw & w(u-v) & v(u-w)\end{vmatrix}$$

by $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1.$

$$=\frac{-2\{(x-y)(x^2-z^2)-(x^2-y^2)(x-z)\}}{v(u-v)(u-w)-w(u-v)(u-w)}$$

$$=\frac{-2(x-y)(x-z)(x+z-x-y)}{(u-v)(u-w)(v-w)}=-2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

EXAMPLES

1. If the Jacobian of u_1 and u_2 with respect to x_1 and x_2 is J_1, and the Jacobian of x_1 and x_2 with respect to u_1 and u_2 is J_2, then prove that $J_1.J_2=1.$ [***Agra*** **07, 10]**

2. Prove that $\quad \dfrac{\partial(u,v,w)}{\partial(x,y,z)}\cdot\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=1.$

[***Indore*** **07,** ***Kanpur*** **09, 11,** ***Bundelkhand*** **12]**

3. If $u^3+v^3+w^3=x+y+z,\ u^2+v^2+w^2=x^3+y^3+z^3,$ $u+v+w=x^2+y^2+z^2$, then prove that

$\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\dfrac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.$ [***Kanpur*** **12,** ***Rohilkhand*** **12]**

4. If $u^3=xyz,\ \dfrac{1}{v}=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z},\ w^2=x^2+y^2+z^2$, prove that

$\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\dfrac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2w(yz+zx+xy)}.$ [***Kumaon*** **09, 13]**

5. If λ,μ,γ are roots of the equation $\dfrac{x}{a+k}+\dfrac{y}{b+k}+\dfrac{z}{c+k}=1$ in k, then find the value of $\dfrac{\partial(x,y,z)}{\partial(\lambda,\mu,\gamma)}=\dfrac{(\lambda-\mu)(\mu-\gamma)(\gamma-\lambda)}{(a-b)(b-c)(c-a)}.$

6. If $u_1=x_1+x_2+x_3+x_4,\ u_1u_2=x_2+x_3+x_4,\ u_1u_2u_3=x_3+x_4,$ $u_1u_2u_3u_4=x_4$, show that $\dfrac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)}=u_1^{\ 3}u_2^{\ 2}u_3.$

[***Kumaon*** **2012]**

13.4. Dependence of functions. *If* $u_1, u_2, \ldots, u_n$ *be functions of n independent variables* $x_1, x_2, \ldots, x_n$, *then a necessary and sufficient condition for them to be connected by a relation* $f(u_1, u_2, \ldots, u_n) = 0$ *is that the Jacobian* $\dfrac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$ *vanishes.*

A necessary condition. If $u_1, u_2, \ldots, u_n$ are not independent functions and there is a relationship amongst them i.e. $f(u_1, u_2, \ldots, u_n) = 0$, then the Jacobian is zero.

Differentiating this relation with respect to $x_1, x_2, \ldots, x_n$, we obtain

$$\frac{\partial f}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1}+\frac{\partial f}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1}+\ldots+\frac{\partial f}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_1}=0$$

$$\frac{\partial f}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2}+\frac{\partial f}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2}+\ldots+\frac{\partial f}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_2}=0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_n}+\frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x_n}+\ldots+\frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x_n}=0$$

Eliminating $\dfrac{\partial f}{\partial u_1}, \dfrac{\partial f}{\partial u_2}, \ldots, \dfrac{\partial f}{\partial u_n}$ from these equations, we get

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \cdots & \dfrac{\partial u_n}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_2}{\partial x_n} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

or
$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = 0.$$

A sufficient condition. If the Jacobian is zero, we will show that there exists a relation amongst the functions $u_1, u_2, \ldots, u_n$ proving that they are not independent.

The equations connecting the functions $u_1, u_2, \ldots, u_n$ and the variables $x_1, x_2, \ldots, x_n$ can be transformed, by elimination, into the following forms:

$$F_1(x_1, x_2, \ldots, x_n, u_1) = 0,$$
$$F_2(x_2, x_3, \ldots, x_n, u_1, u_2) = 0,$$
$$F_3(x_3, x_4, \ldots, x_n, u_1, u_2, u_3) = 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$F_n(x_n, u_1, u_2, \ldots, u_n) = 0.$$

Now, by § 13.31.,

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial F_1}{\partial x_1}\cdot\frac{\partial F_2}{\partial x_2}\cdots\frac{\partial F_n}{\partial x_n} \Bigg/ \frac{\partial F_1}{\partial u_1}\cdot\frac{\partial F_2}{\partial u_2}\cdots\frac{\partial F_n}{\partial u_n}.$$

But we are given that this Jacobian is zero, so that

$$\frac{\partial F_1}{\partial x_1}\cdot\frac{\partial F_2}{\partial x_2}\cdots\frac{\partial F_n}{\partial x_n} = 0.$$

Hence, $\dfrac{\partial F_r}{\partial x_r} = 0$ for some value of r between 1 and n.

This implies that for this particular value of r, the function F_r must not contain x_r, and hence, the corresponding equation for F_r becomes

$$F_r(x_{r+1}, x_{r+2}, \ldots, x_n, u_1, u_2, \ldots, u_r) = 0 \qquad (1)$$

Thus we see that (1) and the remaining equations

$$F_{r+1} = 0, \; F_{r+2} = 0, \; \ldots, \; F_n = 0 \qquad (2)$$

contain $n - r$ independent variables $x_{r+1}, x_{r+2}, \ldots, x_n$. These equations along with (1) are $n - r + 1$ in number.

Thus, eliminating $x_{r+1}, x_{r+2}, \ldots, x_n$ from (1) and (2), we can get a relation connecting $u_1, u_2, \ldots, u_n$ alone.

If the functions are dependent, a relation between them can be found by trial and error.

Ex. 1. Prove that the functions $u=x+y-z, v=x-y+z$, $z=x^2+y^2+z^2-2yz$ are not independent of one another. Also find a relation between them. [*Lucknow* 2010]

We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ 2x & 2(y-z) & 2(z-y)\end{vmatrix}$$

or $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=2\begin{vmatrix}2 & 0 & 0\\ 2 & 0 & 0\\ x & y-z & z-y\end{vmatrix}=0 \qquad \text{by } R_1\to R_2+R_1 \text{ and } R_2\to R_2+R_1$$

This shows that the functions u,v,w are not independent of one another. To find a relation between them, we have

$$u^2=x^2+y^2+z^2+2xy-2yz-2zx$$

and $$v^2=x^2+y^2+z^2-2xy-2yz+2zx \qquad \text{so that}$$

$$u^2+v^2=2(x^2+y^2+z^2-2yz)=2w \qquad \text{which is the}$$

required relation between the functions,

Ex. 2. If $u=x+y+z,\ v=xy+yz+zx,\ w=x^3+y^3+z^3-3xyz$, show that u, v, w are connected by a functional relation and find them. [*Avadh* 2009]

We have $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1 & 1\\ y+z & z+x & x+y\\ 3(x^2-yz) & 3(y^2-zx) & 3(z^2-xy)\end{vmatrix}$$

$$=3\begin{vmatrix}1 & 0 & 0\\ y+z & x-y & x-z\\ x^2-yz & (y-x)(x+y+z) & (z-x)(x+y+z)\end{vmatrix}$$

by $C_2\to C_2-C_1$ and $C_3\to C_3-C_1$

$$=3(y-x)(z-x)\begin{vmatrix}1 & 0 & 0\\ y+z & -1 & -1\\ x^2-yz & x+y+z & x+y+z\end{vmatrix}=0,$$

since C_2 and C_3 are identical.

Hence the functions u, v, w are not independent and must be connected by a functional relation. To find the relation, we have

$$u^3=(x+y+z)^3$$

$$=x^3+y^3+z^3+3(x^2y+x^2z+xy^2+zy^2+xz^2+yz^2)+6xyz$$

$$=\sum x^3+3x(xy+yz+zx)+3y(xy+yz+zx)+3z(xy+yz+zx)-3xyz$$

by adding and subtracting $3xyz$

$$=x^3+y^3+z^3-3xyz+3(x+y+z)(xy+yz+zx)$$

or $u^3=w+3uv.$ which is the required relation.

EXAMPLES

1. Prove that the functions $u=3x+2y-z, v=x-2y+z$, $w=x(x+2y-z)$, are not independent and find a relation between them. [*Lucknow* 2005]

2. If $u=x+2y+z, v=x-2y+3z, w=2xy-zx+4yz-2z^2$, are not independent find a relation between them. [*Jabalpur* 07, *Lucknow* 09]

3. If $u=x^2+y^2+z^2,\ v=x+y+z, w=xy+yz+zx$, show that u,v,w are not independent. Also find the relation between them. [*Meerut* 04, *Raipur* 12 (2nd part)]

4. Prove that the functions $u=x+y-z, v=x-y+z$, $w=x^2+y^2+z^2-2yz$, are not independent. Also find a relation between them. [*Lucknow* 2010]

5. If $u=(x+y)/z,\ v=(y+z)/x,\ w=y(x+y+z)/xz$, prove that u,v,w are not independent.

6. Prove that $ax^2+2hxy+by^2$ and $Ax^2+2Hxy+By^2$ are independent unless $\frac{a}{A}=\frac{h}{H}=\frac{b}{B}$.

ANSWERS

1.	$u^2-v^2=8w.$	**2.**	$u^2-v^2=4w.$
3.	$v^2=u+2w.$	**4.**	$u^2+v^2=2w.$

14

Envelopes, Evolutes

14.1. Families of curves. The equation

$$y = mx + \frac{1}{m}$$

represents, for a given value of m, a straight line. By giving different values to m, we get different straight lines. Suppose now that m continuously varies from one value to another. We get thus a "family" of straight lines.

In general, if $F(x, y, \alpha)$ is an expression involving x, y and α, the curves corresponding to the equation $F(x, y, \alpha) = 0$ constitute a ***family of curves***. If α_1 is a particular value of α, the equation $F(x, y, \alpha_1) = 0$ represents one ***member*** of this family. The symbol α, which is constant for the same member, but different for different members of the family, is called the ***parameter*** of the family.

14.11. Envelope. Definition. A curve which touches each member of a family of curves, and at each point is touched by some member of the family, is called the ***envelope*** of that family of curves.

For instance, we know by Coordinate Geometry that all straight lines whose equation is of the form $y = mx + 1/m$ touch the parabola

$$y^2 = 4x$$

and also that the parabola $y^2 = 4x$ has at every point a tangent which is of the form $y = mx + 1/m$. Hence we infer that the envelope of the family of straight lines $y = mx + 1/m$ is the parabola $y^2 = 4x$.

14.12. Envelope. Another Definition. We can give the following working definition as equivalent to the previous one :

If $F(x, y, \alpha) = 0$ represents a family of curves whose parameter is α, and if the curves $F(x, y, \alpha) = 0$ and $F(x, y, \alpha + h) = 0$ intersect in a point which tends to a definite point P as $h \to 0$, the locus of P (for varying values of α) is called the ***envelope*** of the family.

14.2. Method of finding the Envelope. Let the given family of curves be

$$F(x, y, \alpha) = 0 \qquad (1)$$

Supposing α to have a particular value, this equation represents one member of the family. Let another member of the family be

$$F(x, y, \alpha + h) = 0. \qquad (2)$$

The coordinates of the point of intersection* P_1 of (1) and (2) will satisfy the equation

$$F(x, y, \alpha + h) - F(x, y, \alpha) = 0$$

and, therefore, also the equation

$$\frac{F(x, y, \alpha + h) - F(x, y, \alpha)}{h} = 0.$$

Taking limits as $h \to 0$, we see that the coordinates of the point P to which P_1 tends as $h \to 0$, satisfy the equation

$$\frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0. \qquad (3)$$

Also the coordinates of P must satisfy (1), because P is a point on (1).

If we now eliminate α between (1) and (3), we shall get an equation which the coordinates of P will satisfy for all values of α. That is, the result of eliminating α between (1) and (3) will be the locus of P.

Hence, ***the equation of the envelope of the family of curves $F(x, y, \alpha) = 0$, where α is the parameter, is obtained by eliminating α between the equations***

$$F(x, y, \alpha) = 0 \qquad \textit{and} \qquad \frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0.$$

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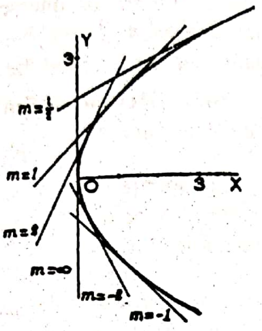
* The argument will apply even when there are more than one points of intersection.

Ex. 1. Find the envelope of the family of straight lines

$y = mx + 1/m.$

[*Gorakhpur* 03, *Agra* 10]



Differentiating partially with respect to the parameter m, we have

$$0 = x - 1/m^2 \quad \text{or} \quad m = x^{-1/2}$$

Substituting this in the first equation, the envelope is given by

$$y = x^{1/2} + x^{1/2} = 2x^{1/2},$$

or $\quad y^2 = 4x,\quad$ a parabola.

A study of the adjoined figure will be instructive in this connection.

Ex. 2. Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = l\sin\alpha\cos\alpha$, where the parameter is α.

[*Bundelkhand* 05, 06, *Avadh* 07, *Indore* 07, *Garhwal* 09, *Rohilkhand* 12, *Raipur* 12, 15]

The given equation can be written as $\dfrac{x}{\sin\alpha} + \dfrac{y}{\cos\alpha} = l \qquad (1)$

Differentiating (1) partially w.r.t. α, we have

$$-\frac{x\cos\alpha}{\sin^2\alpha} + \frac{y\sin\alpha}{\cos^2\alpha} = 0$$

or $\quad \tan^3\alpha = x/y \quad$ i.e. $\quad \tan\alpha = x^{1/3}/y^{1/3} \quad$ whence

$$\sin\alpha = \frac{x^{1/3}}{\sqrt{x^{2/3} + y^{2/3}}} \quad \text{and} \quad \cos\alpha = \frac{y^{1/3}}{\sqrt{x^{2/3} + y^{2/3}}}.$$

Substituting these values in (1), we have

$$\frac{x\sqrt{x^{2/3} + y^{2/3}}}{x^{1/3}} + \frac{y\sqrt{x^{2/3} + y^{2/3}}}{y^{1/3}} = l$$

or $$x^{2/3}\sqrt{x^{2/3} + y^{2/3}} + y^{2/3}\sqrt{x^{2/3} + y^{2/3}} = l$$

or $$(x^{2/3} + y^{2/3})^{3/2} = l \quad \text{i.e.} \quad x^{2/3} + y^{2/3} = l^{2/3}$$

which is an astroid, is the required equation of the envelope.

Ex. 3. Find the envelope of the family of circles $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2$, where α is the parameter.

[*Kumaon* 04, *Indore* 06, *Rohilkhand* 11, *Bundelkhand* 14]

Also interpret the result. [*Meerut* 05, *Avadh* 09]

The given equation is $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha - c^2 = 0. \qquad (1)$

Differentiating (1) partially with respect to α, we get

$$2ax\sin\alpha - 2ay\cos\alpha = 0 \qquad (2)$$

Rearranging the terms of (1), we have

$$2ax\cos\alpha + 2ay\sin\alpha = x^2 + y^2 - c^2 \qquad (3)$$

Squaring (2) and (3), adding and simplifying, we have

$$4a^2(x^2 + y^2) = (x^2 + y^2 - c^2)^2 \qquad (4)$$

which is the required equation of the envelope.

To interpret this result, let us write the equation of the envelope as

$$(x^2 + y^2)^2 - 2c^2(x^2 + y^2) + c^4 = 4a^2(x^2 + y^2)$$

or $$(x^2 + y^2)^2 - 2(2a^2 + c^2)(x^2 + y^2) + c^4 = 0$$

Solving for a quadratic equation in $(x^2 + y^2)$, we get

$$(x^2 + y^2) = \tfrac{1}{2}[2(2a^2 + c^2) \pm \sqrt{4(2a^2 + c^2)^2 - 4c^4}]$$

$$= (2a^2 + c^2) \pm 2a\sqrt{a^2 + c^2}$$

$$= [a^2 + (a^2 + c^2)] \pm 2a\sqrt{a^2 + c^2}$$

$$= [a \pm \sqrt{a^2 + c^2}]^2.$$

This represents two circles of radii $a \pm \sqrt{a^2 + c^2}$ with each having the origin as their centre.

14.3. Finding the Envelope if the Parameter is Quadratic. In case the equation $F(x, y, \alpha) = 0$ is a quadratic in α, say

$$A\alpha^2 + B\alpha + C = 0, \qquad (1)$$

where A, B and C are functions of x and y, the result of differentiation partially with respect to α is

$$2A\alpha + B = 0.$$

Substituting the value of α from this in (1), we have

$$A\left(-\frac{B}{2A}\right)^2+B\left(-\frac{B}{2A}\right)+C=0, \quad \text{or} \quad B^2-4AC=0.$$

Hence ***the envelope of the family of curves***

$$\boldsymbol{A\alpha^2+B\alpha+C=0}$$

where A, B, C are functions of x and y, is

$$\boldsymbol{B^2-4AC=0.}$$

Ex. 1. Find the envelope of the family of straight lines

$y=mx+a/m$. [***Kanpur*** **05,** ***Himachal*** **07,** ***Agra*** **07]**

The equation of the family can be written as $m^2x-my+a=0$, which is a quadratic in m. Hence using the quadratic formula for the parameter where $A=x, B=-y, C=a$, the required envelope is

$$B^2-4AC=0 \quad \text{or} \quad (-y)^2-4.x.a=0.$$

i.e. $y^2=4ax$, which is a parabola.

Ex. 2. Find the envelope of the family of straight lines

$y=mx+\sqrt{a^2m^2+b^2}$, the parameter being m.

[*Rohilkhand* 04, 07, *Bilaspur* 09]

The given equation is $y=mx+\sqrt{a^2m^2+b^2}$

Squaring both sides, $y^2-2mxy+m^2x^2=a^2m^2+b^2$

or $m^2(x^2-a^2)-2mxy+(y^2-b^2)=0$

which is a quadratic equation in the parameter m.

Here $A=x^2-a^2,\ B=-2xy$ and $C=y^2-b^2$. (1)

Hence the envelope is given by $B^2-4AC=0$ (2)

Substituting (1) in (2) and simplifying, we have

$$b^2x^2+a^2y^2-a^2b^2=0$$

or $x^2/a^2+y^2/b^2=1$ which is an ellipse, is the required envelope.

Ex. 3. Find the envelope of the family of curves given by

$\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{k^2-\alpha^2}=1$, where α is the parameter.

[*Jabalpur* 04, *Kanpur* 08, *Rohilkhand* 12]

The given curve is $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{k^2-\alpha^2}=1$

or $x^2(k^2-\alpha^2)+y^2\alpha^2=\alpha^2(k^2-\alpha^2)$

or $\alpha^4-(x^2-y^2+k^2)\alpha^2+k^2x^2=0$

This is a quadratic equation in α^2, so that we have

$$A=1,\ B=-(x^2-y^2+k^2), C=k^2x^2.$$

Then the required envelope is $B^2=4AC$

or $(x^2-y^2+k^2)^2=4k^2x^2$.

EXAMPLES

1. Find the envelope of the family of curves given by
$x^2\sin\alpha+y^2\cos\alpha=a^2$, where α is the parameter.
[*Bundelkhand* 08, *Rohilkhand* 09]

Find the envelope of the following family of straight lines:

2. $y=m^2x+1/m^2$. 3. $y=mx+am^p$, m is the parameter.
4. $(x/a)\cos\theta+(y/b)\sin\theta=1$, where θ is the parameter.
[*Bundelkhand* 09, *Kashi Vidyapeeth* 12]
5. $x/\alpha+y/(a-\alpha)=1$, where α is the parameter.
6. $x\cos\alpha+y\sin\alpha=a$, α being the parameter. **[*Garhwal* 2007]**
7. $x\cos m\theta+y\sin m\theta=a(\cos n\theta)^{m/n}$; where θ is the parameter.

Find the envelope of the following system of circles:

8. $(x-\alpha)^2+y^2=4\alpha$. **[*Agra* 2009]**
9. $(x-\alpha)^2+(y-\alpha)^2=2\alpha$. **[*Lucknow* 09, 12]**

Find the envelope of the following family of curves :

10. $tx^3+t^2y=a$, the parameter being t. **[*Kanpur* 2003]**
11. $x\sec^2\theta+y\,\text{cosec}^2\theta=a$, where θ is the parameter.
12. Find the envelope of the family of straight lines
$ax\sec\alpha-by\,\text{cosec}\,\alpha=a^2-b^2$, where the parameter is α.
[*Purvanchal* 05, *Bilaspur* 06, *Jabalpur* 08, *Raipur* 11, 13, 14]
13. Find the envelope of the family of trajectories:
$y=x\tan\theta-\dfrac{g}{2}\cdot\dfrac{x^2}{u^2\cos^2\theta}$, where θ is the parameter and u, a constant. **[*Lucknow* 2007]**
14. Find the envelope of the ellipses $x=a\sin(\theta-\alpha)$, $y=b\cos\theta$, where α is the parameter. **[*Himachal* 05, *Kanpur* 09, *Agra* 11]**

ANSWERS

1. $x^4+y^4=a^4$. 2. $y^2=4x$.

3. $(p-1)^{p-1}x^p+p^p ay^{p-1}=0$. 4. $x^2/a^2+y^2/b^2=1$.

5. $\sqrt{x}+\sqrt{y}=\sqrt{a}$. 6. $x^2+y^2=a^2$.

7. $r^{n/(m-n)}=a^{n/(m-n)}\cos\{n\theta/(m-n)\}$, where r, θ are the polar coordinates of (x, y).

8. $y^2-4x-4=0$ 9. $(x+y+1)^2=2(x^2+y^2)$.

10. $x^6+4ay=0$. 11. $(x-y)^2-2ax-2ay+a^2=0$.

12. $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

13. $y+\frac{1}{2}gx^2/u^2=u^2/2g$. 14. $x=\pm a$.

14.4. The envelope touches each member of the family. We shall now show that, in general, ***the envelope of a family of curves, deduced on the basis of the second definition, touches each member of the family.***

Let any member of the family be

$$F(x, y, \alpha)=0, \qquad (1)$$

where α is constant and equal to α_1, say.

The equation of the envelope is the result of eliminating α between

$$\left.\begin{aligned} F(x, y, \alpha)&=0,\\ \frac{\partial}{\partial\alpha}F(x, y, \alpha)&=0.\end{aligned}\right\} \qquad (2)$$

Thus the equation of the envelope may be regarded as

$$F(x, y, \alpha)=0 \qquad (3)$$

in which α is not a constant, but a function of x and y given by

$$\partial F(x, y, \alpha)/\partial\alpha=0. \qquad (4)$$

Consider now the point P on (1), where P is the limiting position to which the intersection of $F(x, y, \alpha_1)=0$ and $F(x, y, \alpha_1+h)=0$ tends as $h\to 0$. This point P lies on the curve (1) has the gradient dy/dx given by

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0 \qquad (5)$$

where, while differentiating, α is kept constant and equal to α_1.

But the tangent at P to the envelope has the gradient dy/dx given by

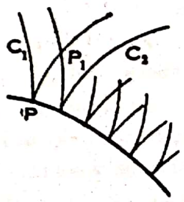
$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}+\left(\frac{\partial F}{\partial\alpha}\cdot\frac{d\alpha}{dx}\right)_{\alpha=\alpha_1}=0 \qquad (6)$$

because α is not constant for the envelope.

But by virtue of equation (4), which is satisfied at every point of the envelope, (6) reduces to (5); i.e. the gradients of the tangents to the curve and the envelope at the common point P are the same. This means that the curve and the envelope have the same tangent at P. In other words, they touch each other at P.

NOTE. If $\partial F/\partial x$ and $\partial F/\partial y$ are both zero, the value of dy/dx cannot be found from equation (5) or (6) and the above argument would break down. So the proposition might not be true for such points. We have seen before (§ 11.4) that if $\partial F/\partial x=\partial F/\partial y=0$ at some point, then there is a singular point there.

14.41. Equivalence of the two definitions. The proposition of the last article enables us to infer at once that, in general, the two definitions of an envelope given in §§ 14.11 and 14.12 would give us the same curve, with the exception that the second definition might in certain cases give us a curve, the whole or a part of which, is not an envelope in the sense of the first definition.



For example, if the curve $F(x, y, \alpha)=0$ is the curve C_1 (which has a cusp at P), and C_2 is the curve $F(x, y, \alpha+h)=0$, it is evident that as $h\to 0$, i.e. as $C_2\to C_1$, P_1 tends to P. Hence the result of eliminating α between $F=0$ and $\partial F/\partial\alpha=0$ will be, or will at least include, the locus of the cusps. But from the figure it is evident that the locus of the cusps will not touch C_1 or C_2 or the other members of the family.

There are other loci (besides the locus of the cusps) which are sometimes obtained in the process of finding the envelope by eliminating α between $F=0$ and $\partial F/\partial\alpha=0$. The consideration of the subject in detail, however, is beyond the scope of the present volume.

14.5. When the Equation of a Family of Curves is not given. If the equation to a family of curves is not given, but the law is given in accordance with which any member of the family can be obtained, then the equation to the family must first be found in a suitable form, whether it be in cartesian or polar form.

Ex. 1. Find the envelope of the circles drawn upon the radii vectors of the ellipse $x^2/a^2+y^2/b^2=1$ as diameter. **[*Nagpur* 2005]**

Any point on the ellipse is $P(a\cos\theta, b\sin\theta)$, θ being the parameter. So, by coordinate geometry, the equation of the circle drawn on the radius vector OP as diameter with end points $(0,0)$ and $(a\cos\theta, b\sin\theta)$ is

$$(x-0)(x-a\cos\theta)+(y-0)(y-b\sin\theta)=0$$

or $$x^2+y^2-ax\cos\theta-by\sin\theta=0. \qquad (1)$$

Differentiating (1) w.r.t. θ, $\quad ax\sin\theta-by\cos\theta=0,$

i.e. $$\tan\theta=by/ax, \quad \text{whence}$$

$$\sin\theta=by/(a^2x^2+b^2y^2)^{1/2},\ \cos\theta=ax/(a^2x^2+b^2y^2)^{1/2} \qquad (2)$$

Substituting (2) in (1) to eliminate θ, the equation of the envelope is

$$x^2+y^2-\frac{a^2x^2}{(a^2x^2+b^2y^2)^{1/2}}-\frac{b^2y^2}{(a^2x^2+b^2y^2)^{1/2}}=0$$

or $$x^2+y^2=(a^2x^2+b^2y^2)^{1/2}$$

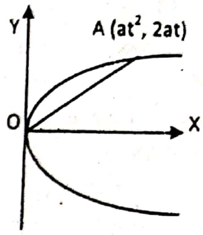
or $$(x^2+y^2)^2=a^2x^2+b^2y^2.$$

Ex. 2. Find the envelope of the circles which pass through the vertex of the parabola $y^2=4ax$ and whose centres lie on it.

Let $A\,(at^2, 2at)$ be any point on $y^2=4ax$. Then the circle with its centre at A which passes through the vertex $O\,(0,0)$ of the parabola, will have the radius $OA=\sqrt{(at^2)^2+(2at)^2}$. Hence, the equation of such a family of circles will be given by

$$(x-at^2)^2+(y-2at)^2=(at^2)^2+(2at)^2$$

or $$x^2+y^2-2axt^2-4ayt=0 \qquad (1)$$



where t is the parameter.

Now, differentiating (1) partially w.r.t. t, we get,

$$-4axt-4ay=0 \quad \text{or} \quad t=-y/x \qquad (2)$$

Eliminating t from (1) and (2), we get the equation of the envelope as

$$x^2+y^2-2ax\left(-\frac{y}{x}\right)^2-4ay\left(-\frac{y}{x}\right)=0$$

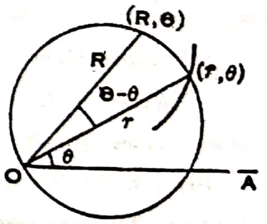
or $$x(x^2+y^2)+2ay^2=0$$

or $$x^3+y^2(x+2a)=0 \quad \text{which is a cissoid.}$$

Ex. 3. Find the envelope of the circles described on the radii vectors of the curve $r^n=a^n\cos n\theta$ as diameter. **[*Lucknow* 2009]**

Let $P(r,\theta)$ be any point on the curve $r^n=a^n\cos n\theta$ and let (R,Θ) be the polar coordinates of any point Q on the circle with the radius vector r (i.e. OP) of the given curve as its diameter.

By geometry, the angle subtended by the diameter of a circle at the circumference is a right angle, so $\angle OQP$ is a right angle. Thus from the right-angled triangle OQP,



$$OQ=OP\cos\angle QOP,$$

i.e. $$R=r\cos(\Theta-\theta),$$

or $$R^n=r^n\cos^n(\Theta-\theta).$$

But $r^n=a^n\cos n\theta$ so that

$$R^n=a^n\cos n\theta\cos^n(\Theta-\theta)$$

or $$R=a\cos^{1/n}n\theta\cos(\Theta-\theta) \qquad (1)$$

Taking the logarithm of both sides of (1), we have

$$\log R=\log a+(1/n)\log\cos n\theta+\log\cos(\Theta-\theta)$$

Differentiating partially w.r.t. θ, which is the parameter,

$$0=\frac{1}{n}\cdot\frac{-n\,n\sin n\theta}{\cos n\theta}+\frac{\sin(\Theta-\theta)}{\cos(\Theta-\theta)}.$$

Hence $\tan(\Theta-\theta)=\tan n\theta.$ or $\Theta-\theta=n\theta+m\pi$ (2)

or $\theta=(\Theta-m\pi)/(n+1).$ (3)

By substituting this value of θ in (1), we get the equation of th envelope as

$$R=a\cos^{1/n}n\left(\frac{\Theta-m\pi}{n+1}\right)\cos\left(\Theta-\frac{\Theta-m\pi}{n+1}\right) \qquad \text{[from (3)]}$$

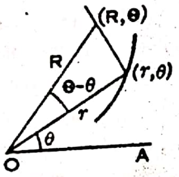
$$=a\cos^{1/n}n\left(\frac{\Theta-m\pi}{n+1}\right)\cos\left(\frac{n\Theta+m\pi}{n+1}\right).$$

Ex. 4. Find the envelope of the straight lines drawn at right angles to the radii vectors of the spiral $r=ae^{\theta\cot\alpha}$ through their extremities.

The equation to the line through $(r,\ \theta)$, at right angles to the radius vector, is

$R\cos(\Theta-\theta)=r,$

i.e. $R\cos(\Theta-\theta)=ae^{\theta\cot\alpha}$ (1)



Differentiating logarithmically w.r.t. the parameter θ, we get

$\tan(\Theta-\theta)=\cot\alpha=\tan(\frac{1}{2}\pi-\alpha).$

Therefore $\Theta-\theta=\frac{1}{2}\pi-\alpha+m\pi,$

or $\theta=\Theta+\alpha-\frac{1}{2}(2m+1)\pi.$

Substituting this in (1), the required envelope is

$$R\cos\{\alpha-\tfrac{1}{2}(2m+1)\pi\}=a\exp\{\Theta+\alpha-\tfrac{1}{2}(2m+1)\pi\}\cot\alpha.$$

which is of the form $R=Ae^{\Theta\cot\alpha}.$

Hence the envelope is another spiral of the same type.

14.6. Two parameters connected by a relation. When the equation of a family of curves contains two parameters connected by a relation, and the elimination of one of them from the equation of the family makes the subsequent process of finding the envelope tedious, we can proceed as in the following example:

Ex. Find the envelope of the straight lines

$$\frac{x}{a}+\frac{y}{b}=1, \qquad (1)$$

where the parameters a and b are related by the equation

$$a^n+b^n=c^n, \quad c \text{ being a constant.} \qquad (2)$$

***[Gorakhpur* 06, *Lucknow* 06, *Rohilkhand* 07, *Bilaspur* 08, *Agra* 09, *Kashi Vidyapeeth* 11]**

Let us regard (for the sake of symmetry) a and b as functions of a single parameter t.

Differentiating (1) with respect to t, we have

$$\frac{x}{a^2}\frac{da}{dt}+\frac{y}{b^2}\frac{db}{dt}=0 \qquad (3)$$

We have to first simplify this. So differentiating (2) with respect to t, we have

$$a^{n-1}\frac{da}{dt}+b^{n-1}\frac{db}{dt}=0 \qquad (4)$$

Equating the values of $\dfrac{da}{dt}\Big/\dfrac{db}{dt}$ obtained from (3) and (4), we have

$$\frac{x}{a^2}\Big/a^{n-1}=\frac{y}{b^2}\Big/b^{n-1} \qquad (5)$$

We have thus to eliminate a and b between (1), (2) and (5). Now (5) gives

$$\frac{x/a}{a^n}=\frac{y/b}{b^n}=\frac{x/a+y/b}{a^n+b^n}=\frac{1}{c^n}. \qquad \text{[using (1) \& (2)]}$$

Hence $a^{n+1}=c^nx,\quad b^{n+1}=c^ny.$

Substituting in (2), we have

$$(c^nx)^{n/(n+1)}+(c^ny)^{n/(n+1)}=c^n$$

or

$$x^{n/(n+1)}+y^{n/(n+1)}=c^{n/(n+1)}$$

which is the equation of the required envelope.

EXAMPLES

1. Find the envelope of the straight lines $x/a+y/b=1$ where the parameters a and b are connected by the following relation, c being a constant :

(i) $a+b=c$. **[*Avadh* 02, *Garhwal* 03]**

(ii) $a^2+b^2=c^2$. [*Bilaspur* 06, *Avadh* 09, *Rohilkhand* 10]

(iii) $a^m b^n=c^{m+n}$. [*Gorakhpur* 2003]

2. Find the envelope of the family of ellipses $x^2/a^2+y^2/b^2=1$ where the parameters a and b are connected by the following relation, c being a constant.:

(i) $a+b=c$. [*Garhwal* 02] (ii) $a^2+b^2=c^2$. [*Avadh* 03]

(iii) $ab=c^2$. [*Lucknow* 2012]

3. If a and b are parameters such that $a^p+b^p=c^p$, then find the envelope of the family of curves $\dfrac{x^m}{a^m}+\dfrac{y^m}{b^m}=1$. [*Rohilkhand* 2012]

4. Show that the envelope of the family of parabolas $(x/a)^{1/2}+(y/b)^{1/2}=1$ under the condition $ab=c^2$ is a hyperbola whose asymptotes coincide with the axes. [*Garhwal* 2008]

5. Find the envelope of the circles which pass through the origin and whose centres lie on the ellipse $x^2/a^2+y^2/b^2=1$. [*Garhwal* 05]

OR Find the envelope of the circles which pass through the centre of the ellipse $x^2/a^2+y^2/b^2=1$ and have their centres upon its circumference. [*Himachal* 2006]

6. Find the envelopes of circles described on the radii vectors of the following curves as diameters:

(i) $y^2=4ax$. (ii) $r^3=a^3\cos 3\theta$.

7. Show that the envelope of the family of circles whose diameters are double ordinates of the parabola $y^2=4ax$ is the parabola $y^2=4a(x+a)$. [*Rajasthan* 03, *Bilaspur* 10, *Kashi Vidyapeeth* 10]

8. Find the envelope of the circle whose diameter is a line of given length which slides between two fixed straight lines at right angles. [*Kashi Vidyapeeth* 2008]

9. Find the envelope of a system of concentric and coaxial ellipses of constant area. [*Avadh* 04, *Gorakhpur* 05, *Kanpur* 10]

10. Find the envelopes of the straight lines, drawn through the extremities of and at right angles to the radii vectors of the following curves:

(i) $r\cos(\theta+\alpha)=p$. (ii) $r=a(1+\cos\theta)$. [*Bilaspur* 2010]

(iii) $r^n\cos n\theta=a^n$. (iv) $r^n=a^n\cos n\theta$.

11. Show that the envelope of circles described on the central radii of a rectangular hyperbola is a lemniscate $r^2=a^2\cos 2\theta$.

12. Show that the envelope of the family of circles described on the double ordinates of the ellipse $x^2/a^2+y^2/b^2=1$ as diameter is the ellipse $\dfrac{x^2}{a^2+b^2}+\dfrac{y^2}{b^2}=1$. [HINT. Take the double ordinates as $(a\cos\theta, b\sin\theta)$ and $(a\cos\theta, -b\sin\theta)$.]

13. Show that the envelope of the polars of points on the ellipse $x^2/h^2+y^2/k^2=1$ with respect to the ellipse $x^2/a^2+y^2/b^2=1$ is $\dfrac{h^2x^2}{a^4}+\dfrac{k^2y^2}{b^4}=1$. [*Bundelkhand* 2011]

ANSWERS

1. (i) $\sqrt{x}+\sqrt{y}=\sqrt{c}$. (ii) $x^{2/3}+y^{2/3}=c^{2/3}$.

(iii) $(m+n)^{m+n}x^m y^n=m^m n^n c^{m+n}$

2. (i) $x^{2/3}+y^{2/3}=c^{2/3}$. (ii) $x\pm y=\pm c$. (iii) $2xy=c^2$.

3. $x^{mp/(m+p)}+y^{mp/(m+p)}=c^{mp/(m+p)}$.

5. $(x^2+y^2)^2=4(a^2x^2+b^2y^2)$.

6. (i) $ay^2+x(x^2+y^2)=0$. (ii) $r^{3/4}=a^{3/4}\cos(3\theta/4)$.

8. A circle $x^2+y^2=c^2$, where c is the length of the given line.

9. $2xy=c^2$, where πc^2 is the given area.

10. (i) $r\cos^2\frac{1}{2}(\theta+\alpha)=p$. (ii) $r=2a\cos\theta$.

(iii) $r^{n/(n+1)}\cos\{n\theta/(n+1)\}=a^{n/(n+1)}$.

(iv) $r^{n/(1-n)}=a^{n/(1-n)}\cos\{n\theta/(n+1)\}$.

14.7. Evolute. ***The locus of the centre of curvature for a curve is called its evolute.***

Ex. 1. Find the evolute of the parabola $y^2=4ax$.

[*Garhwal* 04, *Kanpur* 06, *Purvanchal* 06, *Indore* 04, 08, *Bilaspur* 08, *Kashi Vidyapeeth* 08, *Agra* 06, 09, *Rohilkhand* 10, *Raipur* 12, 14]

Differentiating $y^2 = 4ax$. w.r.t. x, we have

$$\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{\sqrt{4ax}} = \frac{\sqrt{a}}{\sqrt{x}} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\tfrac{1}{2}\sqrt{a}.x^{-3/2}.$$

Let $(\alpha,\ \beta)$ be the coordinates of the centre of curvature of the given parabola, then

$$\alpha = x - \frac{(dy/dx)\{1+(dy/dx)^2\}}{d^2y/dx^2} = x + \frac{2(\sqrt{a}/\sqrt{x})\{1+(a/x)\}}{\sqrt{a}.x^{-3/2}}$$

or $$\alpha = x + 2(x+a) = 3x + 2a. \qquad (1)$$

Also $$\beta = y + \frac{1+(dy/dx)^2}{d^2y/dx^2} = y - \frac{2\{1+(a/x)\}}{\sqrt{a}.x^{-3/2}}$$

or $$\beta = \sqrt{4\,ax} - \frac{2(x+a)\sqrt{x}}{\sqrt{a}} = -2a^{-1/2}\,x^{3/2}. \qquad (2)$$

To obtain the locus of the centre of curvature, we eliminate x between the equations (1) and (2), so that we have

$$\beta^2 = 4a^{-1}x^3 = 4a^{-1}(\alpha - 2a)^3/27,$$

i.e. $$27a\beta^2 = 4(\alpha - 2a)^3.$$

Replacing α and β by x and y respectively, we have as the equation of the evolute

$$27ay^2 = 4(x - 2a)^3.$$

Ex. 2. Find the evolute of the hyperbola $2xy = a^2$.

[*Lucknow* 06, *Bilaspur* 07, *Rohilkhand* 08, *Garhwal* 10, *Bundelkhand* 13, *Raipur* 15]

Differentiating $2xy = a^2$ w.r.t. x, we have

$$2x + 2y\,dy/dx = 0 \quad \text{or} \quad dy/dx = -y/x \qquad (1)$$

and $$\frac{d^2y}{dx^2} = -\frac{x\,dy/dx - y}{x^2} = \frac{2y}{x^2}. \qquad \text{[using (1)]}$$

Let (α, β) be the coordinates of the centre of curvature of the given hyperbola, then

$$\alpha = x - \frac{(dy/dx)\{1+(dy/dx)^2\}}{d^2y/dx^2} = x - \frac{(-y/x)\{1+(-y/x)^2\}}{2y/x^2}$$

or $$\alpha = x + \frac{x^2+y^2}{2x} = \frac{3x^2+y^2}{2x}. \qquad (1)$$

Also $$\beta = y + \frac{1+(dy/dx)^2}{d^2y/dx^2} = y + \frac{\{1+(y/x)^2\}}{2y/x^2}$$

or $$\beta = y + \frac{x^2+y^2}{2y} = \frac{x^2+3y^2}{2y}. \qquad (2)$$

To find the locus of the centre of curvature, we shall eliminate x and y between the equations (1) and (2) We have

$$\alpha + \beta = \frac{3x^2+y^2}{2x} + \frac{x^2+3y^2}{2y} = \frac{3x^2y + y^3 + x^3 + 3xy^2}{2xy} = \frac{(x+y)^3}{2xy}.$$

or $$\alpha + \beta = \frac{(x+y)^3}{2xy}, \quad \text{i.e.} \quad (\alpha+\beta)^{1/3} = \frac{x+y}{(2xy)^{1/3}}$$

Similarly, $$(\alpha - \beta)^{1/3} = \frac{y-x}{(2xy)^{1/3}}$$ so that

$$(\alpha+\beta)^{2/3} - (\alpha-\beta)^{2/3} = \frac{(x+y)^2 - (y-x)^2}{(2xy)^{2/3}} = \frac{4xy}{(2xy)^{2/3}}$$

$$= 2(2xy)^{1/3} = 2(a^2)^{1/3} = 2a^{2/3}. \qquad [\text{since } 2xy = a^2]$$

Replacing α and β by x and y respectively, we have as the equation of the evolute

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$$

14.71. Evolute as the Envelope of the normals. As the centre of curvature of a curve for a given point A on it is the limiting position of the intersection of the normal at A with the normal at any other point B, as $B \to A$, and the evolute is the locus of the centre of curvature, it follows by § 14.12 that ***the evolute of a curve is the envelope of the normals of that curve.***

Therefore (by § 14.12) ***the normals of a curve touch the evolute.*** This an alternative method of finding the evolute.

Ex. 1. Assuming that the evolute of a curve is the envelope of its normals, find the evolute of the parabola $y^2 = 4ax$.

[*Rohilkhand* 02, *Kanpur* 06, *Indore* 04, 08]

Any normal to the parabola with m as the parameter is

$$y = mx - 2am - am^3 \qquad (1)$$

or $$y = m(x - 2a - am^2) \qquad (2)$$

To find the evolute of the parabola, we shall find the envelope of the normal (1). Differentiating (1) w.r.t. m, we have

$$0=x-2a-3am^2 \qquad \text{i.e} \qquad m=\left(\frac{x-2a}{3a}\right)^{1/2}.$$

Substituting this in equation (2) of the normal, we get the evolute

$$y=\left(\frac{x-2a}{3a}\right)^{1/2}\left(x-2a-a\cdot\frac{x-2a}{3a}\right)=\left(\frac{x-2a}{3a}\right)^{1/2}\left(\tfrac{2}{3}(x-2a)\right)$$

or $\qquad 27ay^2=4(x-2a)^3 \qquad$ as before.

Ex. 2. Find the evolute of the ellipse $x^2/a^2+y^2/b^2=1$,

[*Avadh* 06, *Rohilkhand* 04, 07, 11, *Raipur* 11, *Lucknow* 09, 10, 12]

regarding it as the envelope of the normals.

[*Himachal* 07, *Bilaspur* 09]

Any point on the ellipse is given by $(a\cos\theta, b\sin\theta)$. So the equation of a normal to the ellipse at this point can be written as

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2. \qquad (1)$$

where θ is the parameter.

Finding the envelope of the normal (1) will give us the required evolute. So, differentiating (1) partially w.r.t. θ, we have

$$\frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}=0$$

or $\qquad \tan^3\theta=-(by)/(ax) \qquad$ i.e. $\qquad \tan\theta=-(by)^{1/3}/(ax)^{1/3}$

whence $\sin\theta=\dfrac{-(by)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$ and $\cos\theta=\dfrac{(ax)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$.

Substituting these values in (1), we have

$$\frac{ax\sqrt{(ax)^{2/3}+(by)^{2/3}}}{(ax)^{1/3}}+\frac{by\sqrt{(ax)^{2/3}+(by)^{2/3}}}{(by)^{1/3}}=a^2-b^2$$

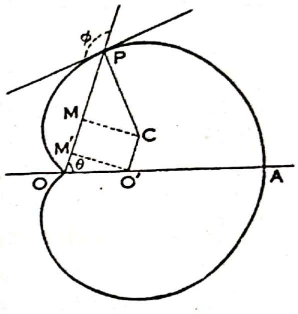
or $\qquad (ax)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}+y^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}=a^2-b^2$

or $\qquad [(ax)^{2/3}+(by)^{2/3}]^{3/2}=a^2-b^2$

or $\qquad (ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3} \qquad$ which is equation of the required evolute.

Ex. 3. Prove that the evolute of the cardioid $r=a(1+\cos\theta)$ is the cardioid $r=\frac{1}{3}a(1-\cos\theta)$, the pole in the latter equation being at the point $(\frac{2}{3}a, 0)$.

Take any point $P(r,\theta)$ on the given cardioid. Let O' be the point $r=\frac{2}{3}a, \theta=0$. Let $C(R,\theta_1)$ be the centre of curvature at P.



Then $O'C=R$, $\angle CO'A=\theta_1$ and $PC=\rho$, the radius of curvature of the given curve $r=a(1+\cos\theta)$ at P.

Let CM and $O'M'$ be perpendiculars dropped on OP. Then

$$\rho=PC=\tfrac{4}{3}a\cos\tfrac{1}{2}\theta.$$

We know that the angle ϕ between the radius vector OP and the tangent at P is connected by the relation

$$\tan\phi=r\frac{d\theta}{dr}=\frac{r}{dr/d\theta}=\frac{a(1+\cos\theta)}{-a\sin\theta}=\frac{2\cos^2\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}=-\cot\tfrac{1}{2}\theta$$

or $\qquad \tan\phi=\tan(\frac{1}{2}\pi+\frac{1}{2}\theta) \qquad$ so $\qquad \phi=\frac{1}{2}\pi+\frac{1}{2}\theta$

and $\qquad \angle CPM=\frac{1}{2}\theta. \qquad$ [from the figure]

From the right-angled triangle CPM, we have

$$CM=PC\sin\tfrac{1}{2}\theta=\tfrac{4}{3}a\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta=\tfrac{2}{3}a\sin\theta.$$

From the right-angled triangle $O'OM'$, we have

$$O'M'=OO'\sin\theta=\tfrac{2}{3}a\sin\theta=CM.$$

Therefore $O'C$ is parallel to OP i.e. $\theta_1=\theta. \qquad (1)$

Also $\quad R=O'C=OP-OM'-MP=r-\frac{2}{3}a\cos\theta-\frac{4}{3}a\cos^2\frac{1}{2}\theta$

since $MP=PC\cos\frac{1}{2}\theta$ from the right-angled triangle CPM

$\qquad =a(1+\cos\theta)-\frac{2}{3}a\cos\theta-\frac{2}{3}a(1+\cos\theta)$

or $\qquad R=\frac{1}{3}a(1-\cos\theta). \qquad (2)$

Substituting (1) in (2), the locus of the centre of curvature C is $R = \frac{1}{3}a(1-\cos\theta_1)$ which is the required evolute.

EXAMPLES

1. Find the evolute of $x^2 - y^2 = a^2$. **[*Bundelkhand* 2012]**

2. Prove that the evolute of the tractrix

$$x = a(\cos t + \log\tan\tfrac{1}{2}t),\ y = a\sin t,$$

is the catenary $y = a\cosh(x/a)$. **[*Bundelkhand* 2010]**

3. Find the equation of the evolute of the hyperbola $x^2/a^2 - y^2/b^2 = 1$. **[*Indore* 2005]**

4. Show that the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$.

[*Agra* 05, *Bilaspur* 06, *Himachal* 06, *Garhwal* 08, *Lucknow* 08]

5. Show that the evolute of a cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is an equal cycloid. **[*Kanpur* 10, *Raipur* 13]**

6. Show that the evolute of an equiangular spiral is an equiangular spiral. **[*Garhwal* 2009]**

[HINT : Start with the pedal equation $p = r\sin\alpha$, where α is a constant, and prove geometrically that if any point on the evolute is (P, R) then $P = R\sin\alpha$, using the fact that $\phi = \alpha$.]

7. Prove that the evolute of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is the envelope of the family of ellipses given by

$$a^2x^2\sec^4\alpha + b^2y^2\operatorname{cosec}^4\alpha = (a^2-b^2)^2,$$

α being the variable parameter. **[*Lucknow* 07, *Kumaon* 09]**

8. Show that the evolute of the curve whose pedal equation is $r^2 - a^2 = mp^2$ is the curve whose pedal equations is $r^2 - (1-m)a^2 = mp^2$.

ANSWERS

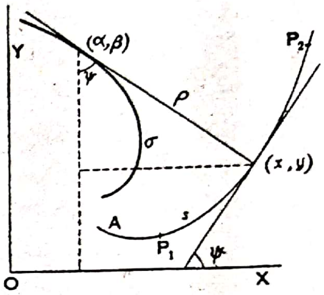
1. $x^{2/3} - y^{2/3} = (2a)^{2/3}$. 3. $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$

14.8. Length of arc of an evolute. ***The difference between the radii of curvature at any two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.***

A suitable sign being given to ρ, it is evident from the figure that, if (α, β) is the centre of curvature corresponding to (x, y) on the curve, then

$$\alpha = x - \rho\sin\psi,\quad \beta = y + \rho\cos\psi \tag{1}$$

Let s be the length of the arc of the given curve measured from some fixed point A on the curve up to (x, y) and σ the length of the arc of the evolute measured from some fixed point on it up to (α, β).



Remembering that

$$dx/ds = \cos\psi,$$

$$dy/ds = \sin\psi,$$

and $d\psi/ds = 1/\rho$,

we have, by differentiating (1) and simplifying,

$$\frac{d\alpha}{ds} = -\frac{d\rho}{ds}\sin\psi,\quad \frac{d\beta}{ds} = \frac{d\rho}{ds}\cos\psi.$$

Squaring, adding, and taking the square root,

$$\frac{d\rho}{ds} = \sqrt{\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2} = \sqrt{1 + \left(\frac{d\beta}{d\alpha}\right)^2}\cdot\frac{d\alpha}{ds}$$

$$= \frac{d\sigma}{d\alpha}\cdot\frac{d\alpha}{ds} = \frac{d\sigma}{ds}.$$

Hence $\dfrac{d\rho}{d\sigma} = 1.$

Therefore $\sigma = \rho + c$, where c is a constant.

Hence $\sigma_2 - \sigma_1 = \rho_2 - \rho_1$, where ρ_1 and ρ_2 are the values of ρ for any two points P_1 and P_2 on the curve, and σ_1 and σ_2 are the corresponding values of σ.

This proves the proposition.

Ex. Prove that the whole length of the evolute of the astroid

$$x = a\cos^3\theta,\ y = a\sin^3\theta \quad \text{is} \quad 12a.$$

The parametric equations of the given astroid are

$$x = a\cos^3\theta, \quad y = a\sin^3\theta \tag{1}$$

Differentiating (1), we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta.\cos\theta}{-3a\cos^2\theta.\sin\theta} = -\tan\theta$$

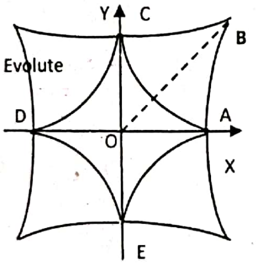
and
$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx} = \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}.$$

The radius of curvature ρ at any point (x, y) of the given astroid is

$$\rho = \frac{(1+dy/dx)^{3/2}}{d^2y/dx} = \frac{(1+\tan^2\theta)^{3/2}}{1/(3a\cos^4\theta\sin\theta)} = \sec^3\theta \cdot 3a\cos^4\theta\sin\theta$$

or
$$\rho = \tfrac{3}{2}a\sin 2\theta.$$

We know that the evolute of an astroid is another astroid whose axes are at an angle of 45° to the axes of the given astroid.



Thus at A, $\theta = 0$, so the radius of curvature is

$$\rho_1 = \tfrac{3}{2}a\sin 0 = 0.$$

At B, $\theta = \pi/4$, so the radius of curvature is

$$\rho_2 = \tfrac{3}{2}a\sin\tfrac{1}{2}\pi = \tfrac{3}{2}a.$$

Hence the length of the arc AB of the evolute is given by

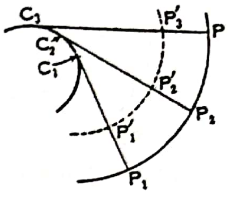
$$\rho_2 - \rho_1 = \tfrac{3}{2}a - 0 = \tfrac{3}{2}a.$$

Since an astroid is symmetric about both the axes, the evolute of (1) being another astroid, will also be symmetric about both the axes.

Hence, the total length of the evolute of the given astroid

$= 8$ times length of the arc AB of the evolute

$= 8 \times (\tfrac{3}{2}a) = 12a.$



14.81. Involutes. If one curve is the evolute of another, then the latter is called an ***involute*** of the former.

Thus if the curve $C_1C_2C_3$ is the evolute of the curve $P_1P_2P_3$, then $P_1P_2P_3$ is an involute of $C_1C_2C_3$.

If C_1 and C_2 are the centres of curvature of the curve $P_1P_2P_3$ at P_1 and P_2 respectively, then by the last article

$$C_1P_1 + \text{arc } C_2C_1 = C_2P_2.$$

Hence, if a thread was wrapped round the curve $C_3C_2C_1$ and was prevented from slipping, it is evident that when the thread is unwrapped, being kept taut all the time, the point on the thread which was at P_1 to begin with will describe the curve $P_1P_2P_3$.

This explains why the curve $C_1C_2C_3$ is called the evolute of the curve $P_1P_2P_3$.

Obviously any point on the thread will describe an involute of the curve $C_1C_2C_3$. Thus ***every curve has an infinite number of involutes.***

If the curves $P_1P_2P_3$ and $P_1'\ P_2'\ P_3'$ are both involutes of the same curve, then they are called parallel curves, because the distance between them measured along their common normal is constant.

EXAMPLES

1. Show that the whole length of the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $4(a^2/b - b^2/a)$.

2. In the evolute of the parabola $y^2 = 4ax$, show that the length of the curve from its cusp $x = 2a$ to the point where it meets the parabola is $2a(3\sqrt{3} - 1)$.

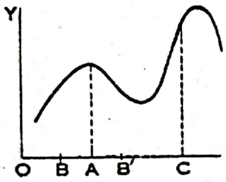
15

Maxima and Minima

15.1. Definition. A function $f(x)$ is said to have a ***maximum*** for a value a of x if $f(a)$ is greater than the value of the function for every other value of x in a small neighbourhood of a, i.e. if an ε can be found such that $f(x)-f(a)$ is negative for all values of x for which

$$0<|x-a|<\varepsilon.$$

Thus if the accompanying figure shows a graph of the function $f(x)$, and $OA = a$, we say that $f(x)$ has a maximum for $x=a$, because $f(a)$ is greater than the values of $f(x)$ for every value of x between B and B' (except, of course, $x=a$ itself, where the inequality reduces to an equality). It will be noticed that, $f(x)$ is said to be a maximum at $x=a$ even though the value of $f(x)$ at $x=a$ is less than the value of $f(x)$ at C. What the definition requires is that $f(x)$ at $x=a$ should be greater than all other values of $f(x)$ in some small neighbourhood. Thus a maximum value of $f(x)$ is not necessarily the greatest value of $f(x)$. In fact, a curve might have several maxima (and minima), also called ***extreme values***.. This is why the term ***local maxima*** and ***local minima*** is used sometimes.



A function $f(x)$ is said to have a ***minimum*** for a value a of x if $f(a)$ is less than the value of the function for every other value of x in a small neighbourhood of a, i.e. if an ε can be found such that $f(x)-f(a)$ is positive for all values of x for which $0<|x-a|<\varepsilon$.

15.2. Conditions for maxima and minima. Let $f(x)$ be a given function of x, and suppose $f(x)$ is expansible in the neighbourhood of $x=a$ by Taylor's theorem.

Then, if $0<\theta<1$,

$$\begin{aligned}f(x) &= f(a)+(x-a)f'(a)+(1/2!)(x-a)^2 f''(a)+\ldots\\ &\quad+\{1/(n-1)!\}(x-a)^{n-1} f^{n-1}(a)\\ &\quad+(1/n!)(x-a)^n f^n\{a+\theta(x-a)\}.\end{aligned}$$

i.e.
$$\begin{aligned}f(x)-f(a) &= (x-a)[f'(a)+(x-a)\{(1/2!)f''(a)\\ &\quad+(1/3!)(x-a)^2 f'''(a)+\ldots\\ &\quad+(1/n!)(x-a)^{n-1} f^n f(a+\theta\cdot\overline{x-a})\}] \qquad (1)\end{aligned}$$

Case I. If $f(x)$ is a maxima at $x=a$, then by definition, $f(x)-f(a)$, and therefore also the right-hand side of (1), must be negative for sufficiently small values of $|x-a|$, whether $x=a$ be positive or negative.

But the series which occurs inside the crooked brackets on the right-hand side of (1) is a finite expression which does not tend to infinity as $x \to a$. Hence the product of this expression and $(x-a)$ can be made as small as we please by making $|x-a|$ sufficiently small. If, therefore, $f'(a) \neq 0$, we can make the product numerically less than $f'(a)$. So for sufficiently small values of $|x-a|$, the sign of the expression within the square brackets in (1) will be the same as that of $f'(a)$.

It follows, therefore, that if $f'(a) \neq 0$, $f(x)-f(a)$ will have one sign when $x-a$ is positive and the opposite sign if $x-a$ is negative. Hence $f(x)$ cannot have a ***maximum*** at $x=a$ if $f'(a) \neq 0$.

Similarly, since for a minimum at $x=a$, $f(x)-f(a)$ must (for sufficiently small values of $|x-a|$) be positive, whether $x-a$ is positive or negative, it follows that $f(x)$ cannot have a *minimum* also at $x=a$ if $f'(a) \neq 0$.

Hence ***a necessary condition that f(x) should have a maximum or a minimum at $x = a$ is that***

$$f'(a) = 0.$$

Case II. If $f'(a) = 0$, then by (1)

$$f(x) - f(a) = (x-a)^2\{(1/2!)f''(a) + (1/3!)(x-a)f'''(a)$$
$$+ \ldots + (1/n!)(x-a)^{n-2} f^n(a + \theta \cdot \overline{x-a})\}.$$

Now the sign of the expression between the crooked brackets is, for sufficiently small values of $|x-a|$, the same as that of $f''(a)$. Also the sign of $(x-a)^2$ is always positive $(x \neq a)$. Hence, if $f''(a)$ is negative, $f(x) - f(a)$ will be negative for sufficiently small values of $|x-a|$, irrespective of whether $x > a$ or $x < a$.

Thus, ***there is a maximum of f(x) at $x = a$ if $f'(a) = 0$ and $f''(a)$ is negative.***

Similarly, ***there is a minimum of f(x) at $x = a$ if $f'(a) = 0$ and $f''(a)$ is positive.***

15.21. Generalisation. The above argument shows also that if $f'(a) = f''(a) = f'''(a) + \ldots = f^{n-1}(a) = 0$, and $f^n(a) \neq 0$, then for a maximum or minimum we must have n even. Also, for a maximum $f^n(a)$ must be negative and for a minimum $f^n(a)$ must be positive.

NOTE. 1. For the validity of this proposition, it is only necessary that $f(x)$ and its first n differential coefficients be continuous at $x = a$. The higher differential coefficients of $f(x)$ may even be non- existent : the propositions will still be true. This becomes evident from (1) of the last article when we notice that $f(x) - f(a)$ now reduces to

$$(1/n!)(x-a)^n f^n\{a + \theta(x-a)\},$$

and the sign of this must, due to the continuity of $f^n(x)$ at $x = a$, be the same as that of $(x-a)^n f^n(a)$ for sufficiently small values of $|x-a|$.

2. The above rule for finding extreme values involving second order derivatives, is also called the **second derivative test.**

15.22. Working Rule. The propositions of § 15.2 and § 15.21 gives us the following rule for determining maxima and minima.

1. *Find $f'(x)$ and equate it to zero. Solve this equation for x. Let the roots be $a_1, a_2, \ldots$.*

2. *Find $f''(x)$ and substitute in it by turns $a_1, a_2, \ldots$. If $f''(a_1)$ is negative, we have a maximum at $x = a_1$. If $f''(a_1)$ is positive, we have a minimum at $x = a_1$.*

3. *If $f''(a_1) = 0$, we must find $f'''(x)$ and substitute in it a_1. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$.*

4. *But if $f'''(a_1)$ is also zero, we must find $f^{iv}(x)$ and substitute in it a_1. If $f^{iv}(a_1)$ is negative, we have a maximum at $x = a_1$, if it is positive, a minimum. If it is zero, we must find $f^v(x)$, and so on.*

Ex. 1. Find the maximum value of $(x-1)(x-2)(x-3)$.

Let $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$.

Then, for a maximum or a minimum, $f'(x) = 0$,

i.e. $$3x^2 - 12x + 11 = 0,$$

whence $$x = \frac{6 \pm \sqrt{36-33}}{3} = 2 \pm 1/\sqrt{3}.$$

Also $$f''(x) = 6x - 12.$$

We see that $f''(2 + 1/\sqrt{3})$ is positive and $f''(2 - 1/\sqrt{3})$ is negative.

Hence $(x-1)(x-2)(x-3)$ has a maximum at $x = 2 - 1/\sqrt{3}$ and a minimum at $x = 2 + 1/\sqrt{3}$.

The maximum value of $f(x) = (x-1)(x-2)(x-3)$ is therefore,

$$(1 - 1/\sqrt{3})(-1/\sqrt{3})(-1 - 1/\sqrt{3}) = (1 - \tfrac{1}{3})(1/\sqrt{3}) = 2/3\sqrt{3}.$$

Ex. 2. Investigate for maximum and minimum values the function given by $y = \sin x + \cos 2x$.

[G.N.D.U. Amritsar 2004]

Since $y' = \cos x - 2\sin 2x = \cos x - 4\sin x \cos x = \cos x(1 - 4\sin x)$ so that $y' = 0$ when $\cos x = 0$ or $\sin x = \frac{1}{4}$.

i.e. when $x = n\pi + \frac{1}{2}\pi$, or $x = \sin^{-1}\frac{1}{4}$, where $n = 0, 1, 2, 3, \ldots$.

Now $y'' = -\sin x - 4\cos 2x = -(-1)^n - 4(-1)$.

For $x = n\pi + \frac{1}{2}\pi$, $y'' > 0$ for every n.

For $x = \sin^{-1}\frac{1}{4}$, $y'' = -\sin x - 4(1 - 2\sin^2 x) = -\frac{1}{4} - 4\{1 - 2(\frac{1}{4})^2\}$.

i.e. y'' is negative.

Hence, we have a minimum at $x = n + \frac{1}{2}\pi$ (n = integer), and a maximum at $x = \sin^{-1}\frac{1}{4}$.

Ex. 3. Find the largest and the smallest values of the polynomial $x^3 - 18x^2 + 96x$ in the interval $[0, 9]$. |*Avadh* 06, *Allahabad* 12|

Let $f(x) = x^3 - 18x^2 + 96x$.

Then, for a maximum or a minimum, $f'(x) = 0$,

i.e. $f'(x) = 3x^2 - 36x + 96 = 0$ or $(x-4)(x-8) = 0$,

i.e. $x = 4, 8$, both of which lie in $[0, 9]$.

Also $f''(x) = 6x - 36$ then $f''(4) = -12 < 0$ and $f''(4) = 12 > 0$. Hence $f(x)$ has a maximum at $x = 4$ and a minimum at $x = 8$.

So the maximum and minimum values of $f(x)$ are $f(4) = 160$ and $f(8) = 128$ respectively. Also the values of $f(x)$ at the end points of $[0, 9]$ are $f(0) = 0$ and $f(9) = 135$.

Thus in $[0, 9]$, the largest and the smallest values of $f(x)$ are to be chosen from $f(4)$, $f(8)$, $f(0)$, $f(9)$ since within the interval, no value is greater than $f(4)$ or smaller than $f(8)$. Also since $[0, 9]$ is a closed interval, its end points also need to be considered.

Hence the largest value of $f(x)$ is 160 and the smallest is 0 in $[0, 9]$.

15.23. Properties of Maxima and Minima. If $f(x)$ is a continuous function and so has as its graph an unbroken curve, a figure will at once show that :

(1) *At least one maximum or one minimum must lie between two equal values of $f(x)$.*

(2) *Maxima and minima occur alternatively.*

(3) *The sign of dy/dx changes from + to − as x passes (while increasing) through the value which makes y a maximum.*

The reverse is the case for a minimum.

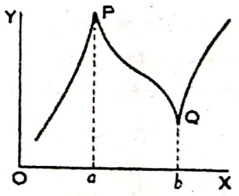
These considerations often enable us to decide, without finding d^2y/dx^2, whether there is a maximum or a minimum at a particular root of $f'(x) = 0$.

15.24. Stationary values. dy/dx is the rate at which y increases, so if dy/dx is zero at $x = a$, we say that y is stationary at $x = a$.

Similarly, d^2y/dx^2 is the rate at which dy/dx increases. So, if d^2y/dx^2 is zero at $x = a$, we say that the tangent to the curve at $x = a$ is stationary.

It is obvious from the above that $y = f(x)$ has a stationary value when it is a maximum or a minimum. At a point of inflexion the tangent is stationary, but we have neither a maximum nor a minimum there even if dy/dx is zero there, i.e. even if the tangent there is parallel to the x-axis.

15.25. Points where the tangent is vertical. At a point like P or Q where we have a cusp with a vertical tangent, dy/dx does not exist because at P the limit of $\{f(x+h) - f(x)\}/h$ on the right is $-\infty$, and its limit on the left is $+\infty$.



Similarly dy/dx does not exist at Q. Hence such points, though they give a maximum or a minimum according to our definition, will not be obtained by the preceding methods. These cases can be treated by § 15.23(3).

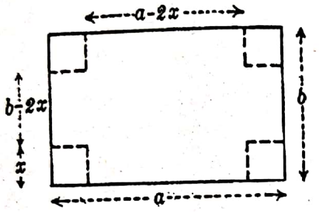
15.3. Applications of Maxima and Minima to Problems. The theory of maxima and minima can be applied in solving problems by finding the first derivative and equating to zero. It is not always necessary to use the second derivative.

Ex. 1. A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is a maximum, the depth will be

$$\tfrac{1}{6}\{(a+b)-(a^2-ab+b^2)^{1/2}\},$$

where a, b are the sides of the original rectangle.

Let x be the length of the side of each of the square portions removed. Let V be the volume of the box thus formed.



So $V=(a-2x)(b-2x)x$ (1)

$$=4x^3-2x^2(a+b)+abx.$$

Therefore, for a maximum or a minimum V, we must have $dV/dx=0$,

i.e. $12x^2-4x(a+b)+ab=0$

or $$x=\frac{2(a+b)\pm\sqrt{4(a+b)^2-12ab}}{12}=\tfrac{1}{6}\{(a+b)\pm(a^2-ab+b^2)^{1/2}\}.$$

As $a^2-ab+b^2=(a-b)^2+ab$, and is therefore positive, the values we have found are real. But the value of x obtained by taking the positive sign before the radical, viz.

$$\tfrac{1}{6}\{(a+b)+(a^2-ab+b^2)^{1/2}\}$$

is (if $a>b$) greater than $\tfrac{1}{6}\{(b+b)+(ab-ab+b^2)^{1/2}\}$, i.e. is greater than $\tfrac{1}{2}b$, which is impossible (see figure).

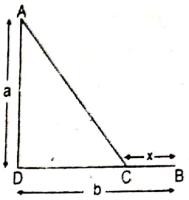
Hence we have a maximum or minimum volume only when

$$x=\tfrac{1}{6}\{(a+b)-(a^2-ab+b^2)^{1/2}\} \tag{2}$$

But $V=0$ when $x=0$, and also when $x=\tfrac{1}{2}b$, and for intermediate values V is positive. Hence the value (2) of x must give a maximum V.

Ex. 2. A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles from that point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec\alpha$. Prove that he should land at a distance $b-a\cot\alpha$ from the place to be reached.

Suppose the person lands at C which is at a distance x from B, the place to be reached. Let A be the point where the boat is stationed and D, the nearest point on the beach.



Then, $AC=\sqrt{a^2+(b-x)^2}.$

We are given, $$\frac{\textit{Rate of Walking}}{\textit{Rate of Rowing}}=\sec\alpha.$$

The time taken in rowing from A to C

$$=\frac{\sqrt{\{a^2+(b-x)^2\}}}{r}$$

where r is the rate of rowing.

The time taken in walking from C to B $=x/r\sec\alpha$.

Hence, total time taken from A to B is

$$T=\frac{\sqrt{\{a^2+(b-x)^2\}}}{r}+\frac{x}{r\sec\alpha}$$

This will be a maximum or minimum when $dT/dx=0$,

i.e. $$\frac{1}{2r}\left\{a^2+(b-x)^2\right\}^{-1/2}\times 2(b-x)(-1)+\frac{1}{r\sec\alpha}=0$$

or $$\frac{(b-x)}{\sqrt{a^2+(b-x)^2}}=\cos\alpha \quad\text{or}\quad (b-x)^2=\{a^2+(b-x)^2\}\cos^2\alpha$$

or $$(b-x)^2\sin^2\alpha=a^2\cos^2\alpha \quad\text{so that}\quad (b-x)=a\cot\alpha$$

or $$x=b-a\cot\alpha$$

Since $$\frac{dT}{dx}=\frac{-(b-x)}{r\sqrt{a^2+(b-x)^2}}+\frac{\cos\alpha}{r}$$

$$\therefore\quad \frac{d^2T}{dx^2}=\frac{1}{r}\left[\frac{\sqrt{a^2+(b-x)^2}-(b-x)^2\left\{a^2+(b-x)^2\right\}^{-1/2}}{\sqrt{a^2+(b-x)^2}}\right]$$

$$=\frac{a^2}{r\left\{a^2+(b-x)^2\right\}^{3/2}}.$$

Hence d^2T/dx^2 is positive for $x=b-a\cot\alpha$, the distance for which time T is a minimum.

15.31. Two variables connected by a relation. If the function whose maxima and minima are to be found is expressed as a function of two variables connected by a given relation, one of the variables should first be eliminated.

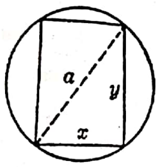
Ex. 3. Assuming that the stiffness of a beam of rectangular cross section varies as the breadth and as the cube of the depth, what must be the breadth of the stiffest beam that can be cut from a log of diameter a?

Let the breadth be x and the depth y, and let S denote the stiffness.

Then $S = k\,xy^3$, where k is a constant.

But $x^2 + y^2 = a^2$.

Hence $S = k\,x(a^2 - x^2)^{3/2}$.



For a maximum or a minimum S, we must have $dS/dx = 0$,

i.e. $(a^2 - x^2)^{3/2} + x.\frac{3}{2}(a^2 - x^2)^{1/2}(-2x) = 0$.

or $a^2 - x^2 - 3x^2 = 0$, i.e. $x = \pm \frac{1}{2}a$.

Hence for the stiffest beam the breadth must be equal to half the diameter of the log.

EXAMPLES

1. Show that $x^5 - 5x^4 + 5x^3 - 1$ has a maximum when $x = 1$, a minimum when $x = 3$, and neither when $x = 0$. [*Kumaon* 2003]

2. Show that $x^3 - 3x^2 + 3x + 7$ has neither a maximum nor a minimum at $x = 1$.

3. If $y = a\log_e x + bx^2 + x$ has extremum values at $x = 1$, $x = 2$, and find a and b. [*Avadh* 2006]

4. Determine the value for which $x/(1 + x\tan x)$ has a maximum value.

5. Find the maximum and minimum values of $(1-x)^2 e^x$.
[*Allahabad* 2012]

6. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$.

7. Show that $\sin x\,(1 + \cos x)$ is a maximum when $x = \frac{1}{3}\pi$.
[*Avadh* 2007]

8. Investigate the maxima and minima of the function $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$ in the interval $0 \le x \le \pi$.

9. Divide a number 15 into two parts such that the square of one multiplied by the cube of the other is a maximum.
[*G.N.D.U. Amritsar* 2004]

10. Prove that the minimum radius vector of the curve $a^2/x^2 + b^2/y^2 = 1$ is of length $a + b$.

11. Show that the triangle of maximum area that can be inscribed in a circle of radius a is an equilateral triangle.
[*Allahabad* 2008]

12. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$. [*Rohilkhand* 2005]

13. Find the volume of the greatest cylinder that can be inscribed in a cone of height h and semi-vertical angle α.

14. A perpendicular is let fall from the centre on a tangent to an ellipse. Show that the maximum value of the intercept between the point of contact and the foot of the perpendicular is $a - b$, where a and b are the semi-axes of the ellipse.

15. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

16. Show that height of a right cone of given capacity and minimum curved surface is $\sqrt{2}$ times the radius of the base. [*Garhwal* 2008]

17. Show that the semi-vertical angle of a right circular cone of given total surface (including area of the base) and maximum volume is $\sin^{-1}\frac{1}{3}$. [*Rohilkhand* 2012]

18. One corner of a long rectangular sheet of paper of width 1 foot is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease.

19. A figure consists of a semicircle with a rectangle on its diameter. Given that the perimeter of the figure is 20 feet, find its dimensions in order that its area may be a maximum.

20. A lane runs at right angles out of a road a feet wide. Find how many feet wide the lane must be if it is just possible to carry a pole b feet long ($b > a$) from the road into the lane, keeping it horizontal.

ANSWERS

3. $a=-2/3, b=-1/6$. 4. $x=\cos x$.

5. Min. value 0 at $x=1$; max. value $4/e$ at $x=-1$.

8. Max. at $x=\pi/4$, $3\pi/4$; min. at $x=2\pi/3$.

9. 9, 6. 13. $\frac{4}{27}\pi h^3 \tan^2 \alpha$. 18. $3\sqrt{3}/4$ ft.

19. Radius of semicircle = height of rectangle $= 20/(4+\pi)$ ft.

20. $b\sin\theta - a\tan\theta$ feet, where $\cos\theta = a^{1/3}/b^{1/3}$.

15.4. Maxima and Minima of functions of two variables. A function $f(x, y)$ of two independent variables x and y, is said to have a ***maximum*** for $x=a$, $y=b$, if $f(a, b)$ is greater than the value of the function for every other pair of values of x, y in a small neighbourhood of a, b; i.e. if

$$f(a, b) > f(a+h, b+k)$$

for all sufficiently small independent values of h and k.

A function $f(x, y)$ is said to have a ***minimum*** for $x=a$, $y=b$, if $f(a, b)$ is less than the value of the function for every other pair of values of x, y in a small neighbourhood of a, b; i.e. if

$$f(a, b) < f(a+h, b+k)$$

for all sufficiently small independent values of h and k.

If $z=f(x,y)$ is regarded as a surface, then at a maximum point, the ordinate to the surface is the largest in the neighbourhood. It corresponds to a 'hill top' from which the surface descends in every direction. Similarly a minimum corresponds to a cup-like depression from which the surface ascends in every direction.

Besides these, we can have a 'ridge' of maximum (or minimum) such that the surface falls (or rises) in all directions except that of the ridge where it remains stationary.

Lastly, we can have a ***'saddle-point'*** on the surface where the tangent plane is horizontal and the surface rises for displacements in some directions and falls for displacements in other directions. All the above cases are classified as ***stationary points.***

15.41. Stationary points. A point (a,b) where the first order partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of the function $f(x, y)$ vanish, is called a ***stationary point*** and its value there is called its ***stationary value.***

A stationary point can be a maximum or minimum or neither of the two.

15.5. Necessary conditions for maxima and minima. To determine the maxima and minima of a function $f(x, y)$ for $x=a$, $y=b$, we investigate the sign of

$$f(a+h, b+k) - f(a, b). \tag{1}$$

If it is the *same* sign for all small values of h and k and is negative, the function has a maximum at (a, b). If it is positive for all small values of h and k, $f(x, y)$ has a minimum.

By Taylor's theorem for functions of two variables

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{\substack{x=a\\ y=b}}$$

$$+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\ y=b}} + \ldots \tag{2}$$

or $$f(a+h, b+k) - f(a, b) = h\, f_x(a, b) + k\, f_y(a, b)$$

since for small values of h and k, the second and higher order terms are much smaller numerically than the first order terms, and may be disregarded when determining the sign of (1). Thus

$$\textbf{sign}\,[f(a+h, b+k) - f(a, b)] = \textbf{sign}\,[hf_x(a, b) + kf_y(a, b)].$$

Taking $k \equiv 0$, we see that the right-hand side changes sign when h changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at $x=a$, $y=b$ if $f_x(a, b) \neq 0$.

Similarly taking $h \equiv 0$, we find that $f(x, y)$ cannot have a maximum or a minimum at $x = a,\ y = b$ if $f_y(a, b) \neq 0$.

Hence ***a set of necessary conditions that $f(x, y)$ should have a maximum or a minimum at $x = a,\ y = b$ is that***

$$f_x(a, b) = 0, \quad f_y(a, b) = 0. \tag{3}$$

These are necessary but not sufficient conditions for the existence of a maximum or a minimum of $f(x, y)$.

If the conditions (3) hold, then by (2),

$$\text{sign}\ [f(a+h, b+k) - f(a, b)] = \text{sign}\ [\tfrac{1}{2}(h^2 r + 2hks + k^2 t)],$$

for small values of h and k. Here, for brevity we have put

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}}, \quad s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a\\y=b}}, \quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}},$$

Now $$h^2 r + 2hks + k^2 t = \frac{1}{r}[h^2 r^2 + 2hkrs + k^2 rt]$$

$$= \frac{1}{r}[(hr + ks)^2 + k^2(rt - s^2)]. \tag{4}$$

The first term inside the brackets is positive. The second will also be positive if $rt > s^2$. In this case the expression (4) will have the same sign for all values of h and k. This sign is determined by the sign of r.

Hence ***another necessary condition that $f(x, y)$ should have a maximum or a minimum at $x = a,\ y = b$ is that***

$$\left[\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right]_{\substack{x=a\\y=b}} > 0 \quad \text{i.e.} \quad rt - s^2 > 0$$

Furthermore, if these conditions are satisfied, ***$f(x, y)$ is a maximum or a minimum according as*** $\left(\dfrac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}}$ ***is*** **negative or positive i.e. $r < 0$ or $r > 0$.**

This is known as **Lagrange's condition** for a maximum or a minimum value of a function of two variables.

If $rt - s^2 < 0$, $f(x,y)$ is not a maximum or a minimum at $x = a, y = b$, then $f(a+h, b+k) - f(a, b)$ has one sign for some values of h, k (e.g. $k = 0, h \neq 0$) while it has another sign for other values ($h = -ks/r, k \neq 0$). Such a point is sometimes called a ***saddle point.***

If $rt - s^2 = 0$, further investigation is needed to determine whether $f(x,y)$ is a maximum at $x = a, y = b$, or not.

15.51. Working rule. The above proposition gives us the following rule for determining the maxima and minima of functions of two variables.

1. *Find $\partial f/\partial x$ and $\partial f/\partial y$ and equate them to zero. Solve these simultaneous equations for x and y. Let $a_1, b_1;\ a_2, b_2; \ldots$ be the pairs of roots.*

2. *Find $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x \partial y$, $\partial^2 f/\partial y^2$, and substitute in them by turns $a_1, b_1;\ a_2, b_2; \ldots$ for x, y. Calculate the value of $rt - s^2$ for each pair of roots.*

3. *If $rt - s^2 > 0$ and $r < 0$ for a pair of roots, $f(x, y)$ is a* ***maximum*** *for this pair. If $rt - s^2 > 0$ and $r > 0$, $f(x, y)$ is a* ***minimum.***

4. *If $rt - s^2 < 0$, the function has neither a maximum nor a minimum but only a* ***saddle point*** *there.*

5. *If $rt - s^2 = 0$, the case is undecided, and further investigation is necessary to decide it.*

When we get a large number of roots, it is more convenient to form a table giving r, s, t and $rt - s^2$ for each pair of roots.

Ex. 1. Discuss the maximum and minimum values of

$$x^3 + y^3 - 3axy.$$

[*Garhwal* 04, *Jabalpur* 05, *Bilaspur* 08, *Kanpur* 08, *Raipur* 11]

The points of maxima and minima are given by

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0, \quad \frac{\partial f}{\partial y} = 3y^2 - 3ax = 0$$

From the first equation above $y=x^2/a$. Substituting in the second,

$$x^4-a^3x=0,\quad \text{or}\quad x(x-a)(x^2+ax+a^2)=0.$$

Hence $x=0$ or a discarding the imaginary roots; then

$$y=x^2/a=0 \text{ or } a.$$

Therefore the function is stationary at $(0, 0)$ and (a, a).

We have $\quad r=\dfrac{\partial^2 f}{\partial x^2}=6x,\ s=\dfrac{\partial^2 f}{\partial x\,\partial y}=-3a,\ t=\dfrac{\partial^2 f}{\partial y^2}=6y$

At $(0, 0)$ $\quad rt-s^2=0-(-3a)^2=-9a^2<0$.

Hence it is a saddle point.

At (a, a), $\quad rt-s^2=6a.6a-(-3a)^2=27a^2$.

Hence (a, a) gives a maximum or a minimum. Since $r=6a$, it is a maximum if a is negative and a minimum if a is positive.

Ex. 2. Show that minimum value of

$$u=xy+(a^3/x)+(a^3/y) \text{ is } 3a^2.$$

[*Rohilkhand* 08, 09, *Lucknow* 10, *Avadh* 11, *Kanpur* 11, *Raipur* 15]

For maximum or minimum, we have

$$\frac{\partial u}{\partial x}=y-\frac{a^3}{x^2}=0,\quad \frac{\partial u}{\partial y}=x-\frac{a^3}{y^2}=0 \quad \text{which give}$$

$$x^2y=a^3 \quad\text{and}\quad xy^2=a^3. \tag{1}$$

Solving these, we get $x^2y=xy^2$ or $xy(x-y)=0$.

This gives $x=0$, $y=0$ and $x=y$.

For $x=0$ and $y=0$, we have from (1), $a=0$ which is not admissible, so we reject these values.

Hence we have $x=y$ so that from (1), we get $x=y=a$.

At $x=y=a$, we have

$$r=\frac{\partial^2 u}{\partial x^2}=\frac{2a^3}{x^3}=\frac{2a^3}{a^3}=2,\ s=\frac{\partial^2 u}{\partial x\,\partial y}=1,\ t=\frac{\partial^2 u}{\partial y^2}=\frac{2a^3}{y^3}=\frac{2a^3}{a^3}=2$$

Therefore $\quad rt-s^2=(2)(2)-1^2=3>0$. Also $\quad r=2>0$.

Hence there is a minima at $x=y=a$.

Thus the minimum value of u

$$=a^2+\frac{a^3}{a}+\frac{a^3}{a}=a^2+a^2+a^2=3a^2.$$

Ex. 3. Determine the points where the function $x^4+y^4-2x^2+4xy-2y^2$ has a maximum or minimum.

[*Rohilkhand* 10, *Kumaon* 13]

We have $\quad \dfrac{\partial f}{\partial x}=4x^3-4x+4y=0,\quad \dfrac{\partial f}{\partial y}=4y^3+4x-4y=0.$

Adding these, we get $4x^3+4y^3=0$, or $x=-y$ discarding the imaginary roots.

Putting $y=-x$ in the first equation, we obtain

$$x^3-2x=0 \text{ i.e. } x=0, \sqrt{2}, \text{ or } -\sqrt{2}.$$

Corresponding values of y are $y=0, -\sqrt{2}$, or $\sqrt{2}$.

We have $\quad r=\dfrac{\partial^2 f}{\partial x^2}=12x^2-4,\ s=\dfrac{\partial^2 f}{\partial x\partial y}=4,\ t=\dfrac{\partial^2 f}{\partial y^2}=12y^2-4.$

At $(0, 0)$, $\quad rt-s^2=(-4)(-4)-4^2=0,$

At $(\pm\sqrt{2}, \mp\sqrt{2})$, $rt-s^2=20.20-4^2>0.$

Since r is positive at $(\pm\sqrt{2}, \mp\sqrt{2})$, hence at these two points the function has a minima. At $(0, 0)$ further investigation is needed.

Now $\quad f(x, y)=x^4+y^4-2(x-y)^2$

$\quad =-2(x-y)^2$ approximately

for small values of x, y. Hence the function is 0 at $(0, 0)$ and negative for all values of x, y, in the neighbourhood except when $x=y$. When $x=y$, the second order terms vanish and $f(x, y)=x^4+y^4=+$ve. Hence $(0,0)$ is a saddle point.

Ex. 4. Discuss the maximum and minimum values of

$$2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)+\cos(x+y).$$

Let $f(x, y)=2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x+y)+\cos^2(x+y)$

Breaking up the first term,

$$f(x, y)=\sin x+\sin y+\cos(x+y)$$

Therefore $\quad \partial f/\partial x=\cos x-\sin(x+y),$

and $\quad \partial f/\partial y=\cos y-\sin(x+y).$

For a maximum or a minimum, $\partial f/\partial x=0=\partial f/\partial y$,

i.e. $\quad \cos x-\sin(x+y)=0=\cos y-\sin(x+y)$

This gives $\cos x = \cos y = \sin(x+y)$

The first set of equations gives $x = 2n\pi \pm y$ (n being an integer). Then the second set gives

$$\cos x = \sin 2x = 2\sin x \cos x \quad \text{or} \quad \cos x\,(1-2\sin x) = 0.$$

Hence, $x = n\pi + \frac{1}{2}\pi$, or $x = \sin^{-1}\frac{1}{2} = n\pi \pm \frac{1}{6}\pi$

Now, $r = \partial^2 f/\partial x^2 = -\sin x - \cos(x+y)$,

$s = \partial^2 f/\partial x\partial y = -\cos(x+y)$,

and $t = \partial^2 f/\partial y^2 = -\sin y - \cos(x+y)$.

For $x = 2m\pi + \frac{1}{2}\pi = y$,

$$r = -1-(-1) = 0,\ s = -(-1) = 1,\ t = -1-(-1) = 0.$$

Hence $rt - s^2 = 0 - 1 = -1 < 0.$

So, there is neither a maximum nor a minimum at $x = m\pi + \frac{1}{2}\pi$.

For $x = (2m+1)\pi + \frac{1}{2}\pi = y$,

$$r = -(-1)-(-1) = 2 = t,\ s = -(-1) = 1.$$

giving $rt - s^2 = 4 - 1 = 3 > 0.$

Also $r > 0$. Hence there is a minima at $x = y = (2m + \frac{3}{2})\pi$

Ex. 5. Discuss the maxima and minima of the function

$u = \sin x \sin y \sin z$, where x, y and z are the angles of a triangle. [*Jabalpur* 04, *Garhwal* 09, *Bundelkhand* 09, 11]

Since $x + y + z = \pi$, (being angles of a triangle), we have,

$$u = \sin x \sin y \sin(\pi - x - y) = \sin x \sin y \sin(x+y).$$

$$\therefore \quad \partial u/\partial x = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y),$$

$$= \sin y\{\cos x \sin(x+y) + \sin x \cos(x+y)\},$$

or $\partial u/\partial x = \sin(2x+y)\sin y.$

Similarly $\partial u/\partial y = \sin(x+2y)\sin x.$

For maxima or minima $\partial u/\partial x = 0 = \partial u/\partial y$ which gives

$$x = y = 0, \quad \text{or} \quad x = y = \tfrac{1}{3}\pi,$$

since we shall only consider values of x, y between 0 and π (being angles of a triangle).

Now $r = \partial^2 u/\partial x^2 = 2\cos(2x+y)\sin y$

$s = \partial^2 u/\partial x\partial y = \sin(x+2y)\cos x + \cos(x+2y)\sin x$

and $t = \partial^2 u/\partial y^2 = 2\cos(x+2y)\sin x$

When $x = y = 0$, we have $r = 0, s = 0, t = 0$ so that $rt - s^2 = 0$.

Hence the case is undecided, and further investigation is needed at $(0,0)$.

When $x = y = \frac{1}{3}\pi$,

$r = 2\cos\pi \sin\frac{1}{3}\pi = -\sqrt{3}$,

$s = \sin(2x+2y) = \sin\frac{4}{3}\pi = -\frac{1}{2}\sqrt{3}$,

and $t = 2\cos\pi\sin\frac{1}{3}\pi = -\sqrt{3}.$

So $rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - (-\sqrt{3}/2)^2 = \frac{9}{4} > 0.$

i.e. $rt - s^2$ is positive and r is negative.

Hence we have a maximum at $x = y = \frac{1}{3}\pi.$

Ex. 6. Find the minimum value of $x^2 + y^2 + z^2$ when $ax + by + cz = p.$

[*Jabalpur* 05, *Lucknow* 06, *Indore* 04, 07, *Avadh* 08, *Bilaspur* 07, *Bundelkhand* 12, 13, *Raipur* 11, 14]

Let $u = x^2 + y^2 + z^2.$ (1)

We shall express u as a function of two independent variables x and y by eliminating z using the given relation, namely,

$$ax + by + cz = p. \tag{2}$$

From (2), $z = (p - ax - by)/c$ which on substitution in (1) gives

$$u = x^2 + y^2 + (p - ax - by)^2/c^2.$$

For maxima or minima, $\partial u/\partial x = 0 = \partial u/\partial y$ which gives

$$\frac{\partial u}{\partial x} = 2x - \frac{2a(p - ax - by)}{c^2} = 0 \quad \text{or} \quad x(a^2 + c^2) + aby - ap = 0. \tag{3}$$

and $$\frac{\partial u}{\partial y} = 2y - \frac{2b(p - ax - by)}{c^2} = 0 \quad \text{or} \quad abx + y(b^2 + c^2) - bp = 0. \tag{4}$$

Multiplying (3) by ab and (4) by $(a^2 + c^2)$ and simplifying, we get

$$yc^2(a^2 + b^2 + c^2) = bpc^2 \quad \text{or} \quad y = bp/(a^2 + b^2 + c^2). \tag{5}$$

Similarly from (3) and (4), we get $x = ap/(a^2 + b^2 + c^2)$. (6)

Now $r=\dfrac{\partial^2 u}{\partial x^2}=2+\dfrac{2a^2}{c^2}$, $s=\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{2ab}{c^2}$, $t=\dfrac{\partial^2 u}{\partial y^2}=2+\dfrac{2b^2}{c^2}$.

So $rt-s^2=4\left(1+\dfrac{a^2}{c^2}\right)\left(1+\dfrac{b^2}{c^2}\right)-\dfrac{4a^2b^2}{c^4}=4\left(1+\dfrac{a^2}{c^2}+\dfrac{b^2}{c^2}\right)$.

Since both $rt-s^2$ and r are positive, u is minimum for the values of x and y given by (5) and (6).

From (2), (5) and (6), we get $z=cp/(a^2+b^2+c^2)$. (7)

Substituting the values of x, y and z given by (5), (6) and (7) in (1), we have the minimum value of u as

$$\frac{(ap)^2+(bp)^2+(cp)^2}{(a^2+b^2+c^2)^2}=\frac{p^2}{a^2+b^2+c^2}.$$

Ex. 7. Find a point within a triangle such that the sum of the square of its distance from the three vertices is a minimum.

[*Garhwal* **05,** *Kumaon* **08,** *Kanpur* **10]**

Let (x, y) be any point within the triangle whose vertices are the points (x_1, y_1), (x_2, y_2) and (x_3, y_3). If u be the sum of the square of the distance of (x, y) from the three vertices, then

$$u=[(x-x_1)^2+(y-y_1)^2]+[(x-x_2)^2+(y-y_2)^2]+[(x-x_3)^2+(y-y_3)^2]$$

For maxima or minima, $\partial u/\partial x=0=\partial u/\partial y$ which gives

$$\frac{\partial u}{\partial x}=2(x-x_1)+2(x-x_2)+2(x-x_3)=0 \quad \text{or} \quad x=\frac{x_1+x_2+x_3}{3},$$

and $$\frac{\partial u}{\partial y}=2(y-y_1)+2(y-y_2)+2(y-y_3)=0 \quad \text{or} \quad y=\frac{y_1+y_2+y_3}{3}.$$

Now $r=\dfrac{\partial^2 u}{\partial x^2}=6$, $s=\dfrac{\partial^2 u}{\partial x\partial y}=0$, $t=\dfrac{\partial^2 u}{\partial y^2}=6$, so that

$$rt-s^2=36>0. \text{ Also } r=6>0.$$

Hence u is a minimum at the point (x, y) i.e. at the point $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$ which is the centroid of the triangle.

EXAMPLES

Discuss the maxima and minima of the following functions

1. $x^2+y^2+6x+12$. [*Garhwal* **07,** *Kashi Vidyapeeth* **10]**

2. $x^2+y^2+2/x+2/y$. [*Bilaspur* **2009]**
3. $y^2+x^2y+x^4$. [*Garhwal* **04,** *Bilaspur* **10]**
4. $x^2y^2-5x^2-8xy-5y^2$.

[*Garhwal* **05,** *Avadh* **06,** *Kumaon* **12,** *Rohilkhand* **12]**

5. $x^2y^2(1-x-y)$. [*Kanpur* **2012]**
6. $2a^2xy-3ax^2y-ay^3-x^3y+xy^3$.
7. $x^4+y^4-2x^2+4xy-2y^2$. [*Rohilkhand* **10,** *Kumaon* **13]**
8. If $u=ax^3y^2-x^4y^2-x^3y^3$, prove that $x=a/2, y=a/3$, makes u a maximum. [*Lucknow* **2009]**
9. Find the maximum and minimum values of the function $u=x^3y^2(1-x-y)$.

[*Purvanchal* **05,** *Jabalpur* **07, 08,** *Kashi Vidyapeeth* **08,** *Garhwal* **10,** *Bundelkhand* **06, 11, 12, 14]**

OR Show that the value of $u=x^3y^2(1-x-y)$ is maximum at $x=1/2, y=1/3$. [*Rohilkhand* **11, 12]**

10. If $u=xy(a-x-y)$, show that the maximum value of u is $a^3/27$. [*Lucknow* **08, 11]**
11. Discuss the maximum or minimum values of u where $u=\sin x\sin y\sin(x+y)$.

[*Indore* **05,** *Lucknow* **05,** *Kumaon* **10, 11,** *Raipur* **12, 14]**

[HINT: Since this function is periodic with period π for both x and y, values of x and y between 0 and π only need be considered]

12. Investigate the maxima and minima of $u=2(x-y)^2-x^4-y^4$ leaving aside any doubtful case that may arise. [*Kanpur* **2009]**
13. Find the minimum distance from the origin to the plane $x+2y-2z-12$. [*Bilaspur* **2009]**
14. Find a point within a triangle such that the sum of the distances from the angular points may be a minimum.
15. A rectangular box is situated in the xy - plane, one corner of this box is at the origin. If the vertex in front of the origin lies on the plane $6x+4y+3z=24$, then find the maximum volume of this box.

[*Jabalpur* **2003]**

ANSWERS

1. Min. at $(-3, 0)$. 2. Max. at $(1, 1)$.
3. Doubtful case at $(0,0)$. 4. Maximum at $(0, 0)$.
5. Max. at $(\frac{2}{5}, \frac{2}{5})$, doubtful case at $(0, 0)$.
6. Max. at $(\frac{1}{2}a, \frac{1}{2}a)$, $(\frac{3}{2}a, -\frac{1}{2}a)$; min. at $(\frac{1}{2}a, -\frac{1}{2}a)$, $(\frac{3}{2}a, \frac{1}{2}a)$.
7. Min. at $(\pm\sqrt{2}, \mp\sqrt{2})$, doubtful case at $(0, 0)$.
8. Maximum at $(\frac{1}{2}, \frac{1}{3})$; ridge of minima for $y = 0, 0 < x < 1$, and ridge of maxima for $y = 0, x < 0$, and $y = 0, x > 1$.
11. Max. at $(\pi/3, \pi/3)$; max. value is $\frac{3\sqrt{3}}{8}$.
12. Max. at $(\pm\sqrt{2}, \mp\sqrt{2})$, undecided at $(0, 0)$. 13. 4.
14. Point is such that each side subtends an angle of 120° there.
15. $64/9$ cube units.

15.6. Lagrange's Multipliers. It is a method of finding the maxima or minima of a function subject to one or more conditions called equality constraints.

Let
$$u = f(x, y, z) \tag{1}$$
be a function of three variables x, y, z, which are connected by a relation
$$\phi(x, y, z) = 0. \tag{2}$$

Only two out of the three variables x, y, z are independent; the third one can be determined in terms of the others by (2). Let us take x and y as the independent variables and z to be dependent on them. Then it follows from § 15.5 that for a stationary point the differential
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{3}$$
must be zero for arbitrary values of dx and dy.

The differential dz is of course dependent on dx and dy. Differentiating (2) we obtain the following relation from which it can be determined:

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0. \tag{4}$$

Multiply (4) by λ and add to (3). Then
$$du = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz.$$

λ is an arbitrary function of x, y, z. Choose it in such a way that
$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0. \tag{5}$$

Then
$$du = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy.$$

Since this is zero for arbitrary dx, dy, we must have
$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}dy = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \tag{6}$$
for a stationary point.

The equations (5) and (6) together with (1) and (2) determine the value of x, y, z and u at the stationary point. This method is known as ***Lagrange's method of undetermined multipliers***. This method is also applicable when we have more than one relation connecting the variables.

While the method is often very convenient in application, the drawback is that it does not determine the nature of the stationary point. This can sometimes be done from physical considerations, as in the examples given below.

Ex. 1. Prove that of all rectangular parallelopipeds of the same volume, the cube has the least surface. **[*Raipur* 2013]**

Let x, y, z be the dimensions of the parallelopiped, V, its volume and S, its surface area.

Then
$$V = xyz, \tag{1}$$
and
$$S = 2xy + 2yz + 2zx, \tag{2}$$

We need to find the minima of S given that V is constant.

Now $dS = 2(y+z)\,dx + 2(x+z)\,dy + 2(x+y)\,dz = 0$

or $(y+z)\,dx + (x+z)\,dy + (x+y)\,dz = 0.$ (3)

Also, from (1), $yz\,dx + zx\,dy + xy\,dz = 0.$ (4)

Multiplying (4) by λ, adding to (3), Lagrange's Method gives

$$(y+z)+\lambda yz = 0, \quad (x+z)+\lambda xz = 0, \quad \text{and} \quad (x+y)+\lambda xy = 0.$$

Multiplying the first equation by x, the second by y and subtracting,

we get $zx - zy = 0$ or $x = y.$

[The root $z = 0$ is discarded, as it will not satisfy (1).]

Similarly, from the second and third equations we get $y = z$. Hence the dimensions of the parallelopiped are

$$x = y = z = V^{1/3}, \qquad \text{[from (1)]}$$

Consider z as the dependant variable and x, y as the independant variables, then from (1), we have

$$yz + xy\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{z}{x}. \tag{5}$$

So $$\frac{\partial S}{\partial x} = 2y + 2y\frac{\partial z}{\partial x} + 2z + 2x\frac{\partial z}{\partial x} = 2y - \frac{2yz}{x}. \qquad \text{[using (5)]}$$

Similarly $$\frac{\partial S}{\partial y} = 2x - \frac{2xz}{y}.$$

Differentiating once more, we get

$$r = \frac{\partial^2 S}{\partial x^2} = \frac{2yz}{x^2} - \frac{2y}{x}\frac{\partial z}{\partial x} = \frac{2yz}{x^2} + \frac{2yz}{x^2} = \frac{4yz}{x^2}. \qquad \text{[using (5)]}$$

$$s = \frac{\partial^2 S}{\partial x \partial y} = 2 - \frac{2z}{y} - \frac{2x}{y}\frac{\partial z}{\partial x} = 2 - \frac{2z}{y} + \frac{2z}{y} = 2,$$

on differentiating $\partial S / \partial y$ partially w.r.t. x and using (5).

Again $$t = \frac{\partial^2 S}{\partial y^2} = \frac{2xz}{y^2} - \frac{2x}{y}\frac{\partial z}{\partial y} = \frac{2xz}{y^2} + \frac{2xz}{y^2} = \frac{4xz}{x^2} \quad \text{since} \quad \frac{\partial z}{\partial y} = -\frac{z}{y}.$$

Hence at $x = y = z$, $r = 4$, $s = 2$ and $t = 4$, so $rt - s^2 = 12 > 0.$

Also since $r > 0$, S is minimum when $x = y = z$ i.e. a parallelopiped of the least surface is a cube.

Ex. 2. Find the stationary values of $x^2 + y^2 + z^2$ subject to

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = 0.$$

Interpret geometrically.

Let $u = x^2 + y^2 + z^2$, then

$$du = 2x\,dx + 2y\,dy + 2z\,dz. \tag{1}$$

Also, from the given conditions,

$$2ax\,dx + 2by\,dy + 2cz\,dz = 0, \tag{2}$$

and $$l\,dx + m\,dy + n\,dz = 0. \tag{3}$$

Multiply (2) by α and (3) by 2μ, and add to (1); then

$$du = 2(x + \lambda ax + \mu l)\,dx + 2(y + \lambda by + \mu m)\,dy + 2(z + \lambda cz + \mu n)\,dz.$$

Equating to zero the coefficients of dx, dy, dz, Lagrange's method gives

$$x + \lambda ax + \mu l = 0, \tag{4}$$

$$y + \lambda by + \mu m = 0, \tag{5}$$

$$z + \lambda cz + \mu n = 0, \tag{6}$$

for a stationary value.

Multiply (4), (5) and (6) by x, y and z respectively and add, then

$$x^2 + y^2 + z^2 + \lambda(ax^2 + by^2 + cz^2) + \mu(lx + my + nz) = 0,$$

or $u + \lambda + 0 = 0.$ or $\lambda = -u.$

Substituting for λ in equations (4) - (6), they become

$$(au-1)x = \mu l, \quad (bu-1)y = \mu m, \quad (cu-1)z = \mu n.$$

To obtain a relation free from μ, we notice that

$$\frac{\mu l^2}{au-1} + \frac{\mu m^2}{bu-1} + \frac{\mu n^2}{cu-1} = lx + my + nz = 0.$$

Hence the stationary values of u are given by the equation

$$\frac{l^2}{au-1} + \frac{m^2}{bu-1} + \frac{n^2}{cu-1} = 0. \tag{7}$$

This is a quadratic in u, giving two stationary values of u.

The surface $ax^2 + by^2 + cz^2 = 1$ is an ellipsoid (or a hyperboloid) and $lx + my + nz = 0$ is a plane. Therefore the point (x, y, z) satisfying both the conditions lies on the conic of their intersection. $x^2 + y^2 + z^2$ gives the square of the distance of (x, y, z) from the origin.

The maximum and minimum values of this distance are the major and minor semi axes of the conic. So equation (7) gives the squares of the lengths of the semi-axes of the conic of intersection.

EXAMPLES

1. Given $f(x, y, z) = 5xyz/(x+2y+4z)$, find the values of x, y, z for which $f(x, y, z)$ is a maximum, subject to the condition $xyz = 8$.

2. If $x + y + z = a$ is given, find the maximum value of xyz. [*Bilaspur* 2010]

3. Find the minimum value of $x^2 + y^2 + z^2$ when $yz + zx + xy = 3a^2$.

4. Divide a into three parts such that their product shall be a maximum.

5. In a plane triangle ABC, find the maximum value of $\cos A \cos B \cos C$. [*Jabalpur* 06, 07, *Lucknow* 07, *Kashi Vidyapeeth* 09]

6. Applying Lagrange's multipliers, show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube. [*Kashi Vidyapeeth* 2010]

7. Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. is $8\,abc/3\sqrt{3}$. [*Himachal* 2005]

8. Show that the maximum and minimum of radii vectors of the section of the surface $(x^2 + y^2 + z^2)^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$ by the plane $\lambda x + \mu y + \nu z = 0$ are given by the equation

$$\frac{a^2\lambda^2}{1-a^2r^2} + \frac{b^2\mu^2}{1-b^2r^2} + \frac{c^2\nu^2}{1-c^2r^2} = 0. \qquad [\textit{Avadh } 2009]$$

ANSWERS

1. 4, 2, 1. **2.** $a^3/27$. **3.** $3a^2$.
4. $a/3, a/3, a/3$. **5.** $1/8$.

16

Indeterminate Forms.

16.1. Indeterminate Forms. The limit of $\phi(x)/\psi(x)$ as $x \to a$ is, in general, equal to the limit of the numerator divided by the limit of the denominator. But when these two limits are both zero, the quotient reduces to the form 0/0, which is meaningless.

This does not imply that $\lim_{x\to a}\{\phi(x)/\psi(x)\}$ is meaningless, or that it does not exist; the only conclusion which we may draw is that the method adopted by us is unsuitable. In the present chapter we shall consider what should be done in such and similar other cases.

The form 0/0 is called an ***indeterminate form***. Other indeterminate forms are ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞ and ∞^0.

From what follows it should be clear that we do not find the *value* of 0/0, or any of the other indeterminate forms. We find the *limits* of the combinations of functions whose limits assume these forms when we take the limits of the functions separately.

16.11. The form 0/0. Let $\phi(x)$ and $\psi(x)$ be functions which are expansible by Taylor's Theorem in the neighbourhood of $x = a$. Also let $\phi(a) = 0$, and $\psi(a) = 0$.

Then $$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

We have by Taylor's Theorem, $\lim_{x\to a}\phi(x)/\psi(x) =$

$$\lim_{x\to a}\frac{\phi(a)+(x-a)\phi'(a)+(1/2!)(x-a)^2\phi''(a)+\ldots+R_1}{\psi(a)+(x-a)\psi'(a)+(1/2!)(x-a)^2\psi''(a)+\ldots+R_2}$$

Here $R_1 = (1/n!)(x-a)^n \phi^{(n)}\{a+\theta(x-a)\}, \quad 0<\theta<1,$ and R_2 has a similar meaning.

Now, since $\phi(a)=0,$ and $\psi(a)=0,$ we get, after dividing both numerator and denominator by $(x-a)$,

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(a)+(x-a)\{(1/2!)\phi''(a)+\ldots\}}{\psi'(a)+(x-a)\{(1/2!)\psi''(a)+\ldots\}}$$

$$=\frac{\phi'(a)}{\psi'(a)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

This proves the proposition which is generally called **L'Hopital's Rule.**

It is easy to see that if $\phi'(a), \phi''(a), \ldots\ldots, \phi^{(n-1)}(a),$ and $\psi'(a), \psi''(a), \ldots\ldots, \psi^{(n-1)}(a),$ are all zero, but $\phi^{(n)}(a)$ and $\psi^{(n)}(a)$ are not both zero, then

$$\boxed{\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi^{(n)}(x)}{\psi^{(n)}(x)}.}$$

NOTE 1. The existence of differential coefficients of order higher than n need not be assumed.

2. The proposition of this article is true even when x tends to ∞ instead of a. For writing $x = 1/t$, we have, if

$$\lim_{x\to\infty}\phi(x)=0 \quad\text{and}\quad \lim_{x\to\infty}\psi(x)=0.$$

then $$\lim_{x\to\infty}\frac{\phi(x)}{\psi(x)} = \lim_{t\to 0}\frac{\phi(1/t)}{\psi(1/t)}$$

$$=\lim_{t\to 0}\frac{\phi'(1/t).t^{-2}}{\psi'(1/t).t^{-2}},$$ by the proposition proved above,

$$=\lim_{t\to 0}\frac{\phi'(1/t)}{\psi'(1/t)} = \lim_{x\to\infty}\frac{\phi'(x)}{\psi'(x)}.$$

IMPORTANT. The student must not differentiate $\phi(x)/\psi(x)$ as a fraction by the quotient rule of differentiation. The numerator and denominator have to be differentiated separately.

Sometimes it is necessary to use L' Hopital's Rule several times in the same problem as can be seen in the following examples. The use of standard results on limits, such as

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \quad\text{and}\quad \lim_{x\to 0}\frac{\tan x}{x}=1$$

can simplify and shorten the process of continuously differentiating the numerator and denominator. The student should employ such methods whenever possible.

Ex. 1. Evaluate $\lim_{x\to 0}(a^x - b^x)/x.$ [*Agra* **03,** *Indore* **04, 07**]

We have $$\lim_{x\to 0}\frac{a^x-b^x}{x} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{a^x\log a - b^x\log b}{1} = \log a - \log b = \log\frac{a}{b}.$$

Ex. 2. Evaluate $\lim_{x\to 0}\dfrac{\tan^2 x - x^2}{x^2\tan^2 x}.$ [*Delhi* **2008**]

We have $$\lim_{x\to 0}\frac{\tan^2 x-x^2}{x^2\tan^2 x} = \lim_{x\to 0}\frac{\tan^2 x - x^2}{x^4}.\lim_{x\to 0}\frac{x^2}{\tan^2 x}$$

$$=\lim_{x\to 0}\frac{\tan^2 x - x^2}{x^4}\left(\frac{0}{0}\right) \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$=\lim_{x\to 0}\frac{2\tan x\sec^2 x - 2x}{4x^3} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2\sec^4 x + 2\tan x\cdot 2\sec x\cdot\sec x\tan x - 2}{12x^2}$$

$$=\lim_{x\to 0}\frac{\sec^4 x + 2\tan^2 x\sec^2 x - 1}{6x^2}$$

$$=\lim_{x\to 0}\frac{3\sec^4 x - 2\sec^2 x - 1}{6x^2} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{12\sec^4 x\tan x - 4\sec^2 x\tan x}{12x}$$

$$=\lim_{x\to 0}\frac{12\sec^4 x - 4\sec^2 x}{12}\cdot\lim_{x\to 0}\frac{\tan x}{x} = \frac{12-4}{12}\cdot 1 = \frac{2}{3}.$$

Ex. 3. If the limit of $\dfrac{\sin 2x - a\sin x}{x^3}$ as $x \to 0$, be finite, find the value of a and the limit. **[*GNDU Amritsar* 04, *Delhi* 06]**

We have
$$\lim_{x\to 0} \frac{\sin 2x - a\sin x}{x^3} = \lim_{x\to 0} \frac{2\sin x\cos x - a\sin x}{x^3}$$
$$= \lim_{x\to 0} \frac{\sin x\,(2\cos x - a)}{x^3}$$
$$= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{(2\cos x - a)}{x^2}$$
$$= \lim_{x\to 0} \frac{(2\cos x - a)}{x^2} \qquad \left(\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right) \qquad (1)$$

From (1), we see that $\lim_{x\to 0} x^2 = 0$. For the above fraction to have a finite limit, the numerator must also be zero as $x \to 0$,

i.e. $\lim_{x\to 0}(2\cos x - a) = 0$ or $2 - a = 0$ or $a = 2$ (2)

From (1) and (2), we have
$$\lim_{x\to 0} \frac{2\cos x - 2}{x^2} \qquad \left(\frac{0}{0}\right)$$
$$= \lim_{x\to 0} \frac{-2\sin x}{2x} \qquad \left(\frac{0}{0}\right)$$
$$= -\lim_{x\to 0} \frac{\sin x}{x} = -1.$$

Hence, $a = 2$ and the limit is -1.

16.12. Alternative proof. We can deduce **De l' Hopital's Rule** (§ 16.11) very easily from Cauchy's theorem. The rule itself can now be stated in a more general form as follows:

Let $\phi(x)$ and $\psi(x)$ be continuous in $a \le x \le b$, and let $\phi'(x)$ and $\psi'(x)$ exist in $a < x < b$; further, let $\phi(a) = 0$, $\psi(a) = 0$ and $\psi(x) \ne 0$ in $a < x < b$. Then

$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)} \quad \textit{if the latter limit exists.}$$

For values of x less than b, we have
$$\frac{\phi(x)}{\psi(x)} = \frac{\phi(x) - \phi(a)}{\psi(x) - \psi(a)}, \text{ since } \phi(a) = \psi(a) = 0,$$

$$= \frac{\phi'(x_1)}{\psi'(x_1)}, \quad a < x_1 < x, \text{ by Cauchy's theorem.}$$

Taking limits, and remembering that $x_1 \to a$ as $x \to a$, we get,
$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)}.$$

It is to be noticed that b may be very near to a if necessary. Also, since $a < x$, the above limits are really right-hand limits. It is evident, however, that we can enunciate the rule to cover the case when $x \to a$ from the left.

The advantage of deriving the rule from Cauchy's theorem lies in the fact that the existence of the derivatives at $x = a$ is not necessary.

If $\phi'(x)/\psi'(x)$ assumes the form $0/0$, we can apply the rule again.

16.2. Algebraic Methods. In cases where the expansions of the functions involved are known, or some of the limits are known, they may be used either to solve the problem or to shorten the work.

The student should note that differentiating the numerator or the denominator repeatedly amounts really to finding the coefficients in the expansion of the function concerned, by Maclaurin's Theorem; and if the expansion is already known, the work can be shortened.

But in some cases, when on account of the non-existence of the differential coefficients, Taylor's and Maclaurin's theorems are not applicable, we can still get the result by algebraic methods.

Ex. 1. Find $\lim_{x\to 0} \dfrac{x^{1/2}\tan x}{(e^x - 1)^{3/2}}$. **[*Rohilkhand* 05, *Kanpur* 07]**

As $x^{1/2}$ cannot be expanded in a Taylor's series in the neighbourhood of $x = 0$, L' Hopital's Rule is not applicable.

But
$$\lim_{x\to 0} \frac{x^{1/2}\tan x}{(e^x - 1)^{3/2}} = \lim_{x\to 0} \frac{x^{1/2}\tan x}{(x + x^2/2! + \dots)^{3/2}},$$
by the Exponential Theorem

$$= \lim_{x\to 0} \frac{x^{1/2} \tan x}{x^{3/2}(1+x/2!+\dots)^{3/2}}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \frac{1}{(1+x/2!+\dots)^{3/2}} = 1 \quad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

Ex. 2. Evaluate $\lim_{x\to 0} \frac{(1+x)^{1/x}-e}{x}$. ✓

[*Kumaon* 04, *Indore* 05, *Rohilkhand* 07, *Magadh* 01, 08, *Patna* 02, 04, 08, *Allahabad* 08]

We have $\lim_{x\to 0} \frac{(1+x)^{1/x}-e}{x}$ $\quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}(1+x)^{1/x}}{1} = \lim_{x\to 0} \frac{d}{dx}(1+x)^{1/x} \qquad (1)$$

Let $y=(1+x)^{1/x}$ so that $\log y = \frac{1}{x} \log (1+x)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{(1+x)} - \frac{1}{x^2} \log (1+x)$$

$$= \frac{1}{x}(1+x)^{-1} - \frac{1}{x^2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)$$

$$= \frac{1}{x}(1-x+x^2-\dots) - \left(\frac{1}{x} - \frac{1}{2} + \frac{1}{3}x\dots\right)$$

$$= (-1+x-\dots) + \tfrac{1}{2} - (\tfrac{1}{3}x - \dots)$$

Then $\frac{dy}{dx} = y\left(-\frac{1}{2} + \frac{2}{3}x + \text{terms with higher powers of } x\right)$

or $\frac{d}{dx}(1+x)^{1/x} = (1+x)^{1/x}\left(-\frac{1}{2} + \frac{2}{3}x + \text{terms of higher powers of x}\right)$

or $\lim_{x\to 0} \frac{d}{dx}(1+x)^{1/x} = \lim_{x\to 0}(1+x)^{1/x} \cdot \left(-\frac{1}{2}+0\right) = e\,(-\tfrac{1}{2}) = -\tfrac{1}{2}e$

since we know that $\lim_{x\to 0}(1+x)^{1/x} = e$.

Substitutng this value in (1), we get $\lim_{x\to 0} \frac{(1+x)^{1/x-e}}{x} = -\frac{1}{2}e$.

EXAMPLES

Evaluate the following limits :

1. $\lim_{x\to 0}(e^{ax}-e^{-ax})/\log(1+bx)$. 2. $\lim_{x\to 0}(x-\sin x)/x^3$.

3. $\lim_{x\to 0}(x-\tan x)/x^3$. **[*Bundelkhand* 06, *Kashi V.* 10]**

4. $\lim_{x\to 0}(e^x-e^{\sin x})/(x-\sin x)$ **[*Magadh* 03, *Patna* 03]**

5. $\lim_{x\to 0}\{xe^x-\log(1+x)\}/x^2$. **[*Rohilkhand* 09, *Agra* 06, 10]**

6. $\lim_{x\to 0}\log(1-x^2)/\log\cos x$.

7. $\lim_{x\to 0}(\tan x-x)/x^2\tan x$.

8. $\lim_{x\to 0}(\tan x-\sin x)/x^3$. **[*Lucknow* 07, 10]**

9. $\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$. **[*Meerut* 2005]**

10. $\lim_{x\to 0}\frac{e^x+\log(1-x)-1}{\tan x-x}$. 11. $\lim_{x\to 0}\frac{a^x-x^a}{x^x-a^a}$.

[*Rohilkhand* 2003]

12. $\lim_{x\to 1}\frac{x^x-x}{1-x+\log x}$. 13. $\lim_{x\to 0}\frac{\sin x-x+\frac{1}{6}x^3}{x^5}$.

[*Allahabad* 2004] **[*Kumaon* 2005]**

14. $\lim_{x\to 0}\frac{e^x-e^{x\cos x}}{x-\sin x}$. **[*Kumaon* 2012]**

15. Prove that $\lim_{x\to 0}\frac{\sin x \,.\, \sin^{-1}x-x^2}{x^6}=\frac{1}{18}$. **[*Indore* 2006]**

16. Find the values of a, b, c so that

$\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2$ **[*Kashi V.* 09, *Lucknow* 09]**

17. Find the values of a and b in order that

$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$ may be equal to 1.

[*Allahabad* 02, *Kashi Vidyapeeth* 08]

18. Prove that $\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}=\frac{11e}{24}$. **[*Purvanchal* 05, *Avadh* 06, *Kashi Vidyapeeth* 09, *Patna* 10, *Kanpur* 12]**

ANSWERS

1. $2a/b$. **2.** $\frac{1}{6}$. **3.** $-\frac{1}{3}$. **4.** 1.

5. $\frac{3}{2}$. **6.** 2. **7.** $\frac{1}{3}$. **8.** $\frac{1}{2}$.

9. -1. **10.** $-\frac{1}{2}$. **11.** $(\log a-1)/(\log a+1)$.

12. -2. **13.** $\frac{1}{120}$. **14.** 3.

16. $a=1, b=2, c=1$. **17.** $a=-\frac{5}{2},-\frac{3}{2}$.

16.3. The Form ∞/∞. *Let* $\lim_{x\to a}\phi(x)$ *and* $\lim_{x\to a}\psi(x)$ *be both infinite. Then*

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

For,
$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{1/\psi(x)}{1/\phi(x)},$$

which is of the form $0/0$.

$$=\lim_{x\to a}-\frac{\psi'(x)}{\{\psi(x)\}^2}\Big/-\frac{\phi'(x)}{\{\phi(x)\}^2}, \quad \text{by §16.11,}$$

$$=\lim_{x\to a}\left\{\frac{\psi'(x)}{\phi'(x)}\cdot\left(\frac{\phi(x)}{\psi(x)}\right)^2\right\}.$$

Thus
$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\psi'(x)}{\phi'(x)}\cdot\left\{\lim_{x\to a}\frac{\phi(x)}{\psi(x)}\right\}^2. \quad (1)$$

Now let $\lim_{x\to a}\{\phi(x)/\psi(x)\}=\lambda$ (2)

Three cases arise.

CASE I. $\lambda\neq 0, \neq\infty$. In this case, we can divide both sides of equation (1) by λ^2 and thus obtain the following:

$$\lambda^{-1}=\lim_{x\to a}\frac{\psi'(x)}{\phi'(x)}.$$

Hence
$$\lambda=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

CASE II. $\lambda=0$. Adding 1 to each side of equation (2),

$$\lambda+1=\lim_{x\to a}\frac{\phi(x)+\psi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)+\psi'(x)}{\psi'(x)}.$$

by Case I.

$$=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}+1.$$

Hence
$$\lambda=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

CASE III. $\lambda=\infty$. Here

$$\frac{1}{\lim_{x\to a}\{\phi(x)/\psi(x)\}}=\lim_{x\to a}\frac{\psi(x)}{\phi(x)}=\lim_{x\to a}\frac{\psi'(x)}{\phi'(x)},$$

by Case II.

Therefore
$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

Hence in every case in which $\lim_{x\to a}\phi(x)=\infty$ and $\lim_{x\to a}\psi(x)=\infty$, we have

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

Although the quotient of two functions can at once be put in a form which would give rise to the indeterminate form $0/0$, instead of ∞/∞, and vice versa, care should be taken to select the form which would enable us to evaluate the limit most quickly. Also, in most cases coming under the form ∞/∞, it is necessary to transform to the form $0/0$ as soon as convenient; otherwise the process of differentiating the numerator and denominator would never terminate. Thus, if we once have x^{-1} in the numerator (or the denominator) and the limit when $x\to 0$ has to be found, successive differentiations would involve x^{-2}, x^{-3}, x^{-4},....., which all $\to\infty$ (numerically) as $x\to 0$. So a change to the form $0/0$ has to be made at a suitable stage.

NOTE. 1. As before, by writing $x=1/t$, we can show that the proposition of this article is true even when $x\to\infty$.

2. The proposition is evidently true also when one or both the limits are $-\infty$.

Ex. Find $\lim_{x\to 0} \dfrac{\log x}{\cot x}$. [*Agra* 2006]

This is of the form ∞/∞. We have, therefore,

$$\lim_{x\to 0}\frac{\log x}{\cot x}=\lim_{x\to 0}\frac{1/x}{-\operatorname{cosec}^2 x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{-\sin^2 x}{x} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-2\sin x\cos x}{1}=0.$$

16.31. The form $0\times\infty$. This can easily be reduced to the form 0/0 or to the form ∞/∞. Let $\lim_{x\to a}\phi(x)=0$ and $\lim_{x\to a}\psi(x)=\infty$, then

$$\lim_{x\to a}\phi(x).\psi(x)=\lim_{x\to a}\frac{\phi(x)}{1/\psi(x)} \qquad \left(\frac{0}{0}\right)$$

or $$\lim_{x\to a}\phi(x).\psi(x)=\lim_{x\to a}\frac{\psi(x)}{1/\phi(x)} \qquad \left(\frac{\infty}{\infty}\right)$$

which can now be evaluated by the methods given before.

Ex. Evaluate $\lim_{x\to 0} x\log x$.

We have $\lim_{x\to 0} x\log x$. (which is of the form $0\times\infty$)

$$=\lim_{x\to 0}\frac{\log x}{1/x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{1/x}{-1/x^2}=\lim_{x\to 0}(-x)=0.$$

EXAMPLES

Evaluate the following limits:

1. $\lim_{x\to 0}\dfrac{\log\sin 2x}{\log\sin x}$ [*Jabalpur* 2006]
2. $\lim_{x\to 0}\dfrac{\operatorname{cosec} x}{\log x}$. [*Kashi Vidyapeeth* 08, *Agra* 09, *Kumaon* 13]
3. $\lim_{x\to\infty}\dfrac{\log x}{a^x}$ $a>1$ 4. $\lim_{x\to\infty}\dfrac{e^x}{x^3}$ [*Jabalpur* 2007]

5. $\lim_{x\to 1}\dfrac{\log(1-x)}{\cot\pi x}$ [*Agra* 05, *Lucknow* 10]
6. $\lim_{x\to 1}\dfrac{\log(x-1)+\tan\frac{1}{2}\pi x}{\cot\pi x}$, $x\to 1$ from the right.
7. $\lim_{x\to 0}\log_{\sin x}(\sin 2x)$

[Hint: Change the base so that $\log_{\sin x}(\sin 2x)=\dfrac{\log_e\sin 2x}{\log_e\sin x}$.]

8. $\lim_{x\to\frac{1}{2}}\dfrac{\sec\pi x}{\tan 3\pi x}$ 9. $\lim_{x\to 0}\sin x\log x$

⑩ $\lim_{x\to 0}\dfrac{\cancel{\log}\log(1-x^2)}{\cancel{\log}\log\cos x}$ [*Jabalpur* 2004]

11. $\lim_{x\to\infty}2^x\sin(a/2^x)$ [*Agra* 03] 12. $\lim_{x\to\infty}x(a^{1/x}-1)$
13. $\lim_{x\to 1}\sec\dfrac{\pi}{2x}\cdot\log x$ [*Lucknow* 10, *Allahabad* 12]
14. $\lim_{x\to\infty}x^m e^{-x}$, where m is a positive integer. [*Agra* 2007]

ANSWERS

1. 1. **2.** $-\infty$. **3.** 0. **4.** ∞.
5. 0. **6.** -2. **7.** 1. **8.** 3.
9. 0. **10.** $\frac{1}{2}$. **11.** a. **12.** $\log a$.
13. $2/\pi$. **14.** 0.

16.32. The form $\infty-\infty$. This can be reduced to the form 0/0 or ∞/∞. Let $\lim_{x\to a}\phi(x)=\infty$ and $\lim_{x\to a}\psi(x)=\infty$, then

$$\lim_{x\to a}\{\phi(x)-\psi(x)\}=\lim_{x\to a}\frac{1/\psi(x)-1/\phi(x)}{1/\psi(x)\cdot\phi(x)}$$

and if the left-hand side is of the form $\infty-\infty$, the right-hand side is of the form 0/0 .

Ex. 1. Evaluate $\lim_{x\to\pi/2}(\sec x-\tan x)$.

$$\lim_{x\to\pi/2}(\sec x-\tan x)=\lim_{x\to\pi/2}\frac{1-\sin x}{\cos x} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to\pi/2}\frac{-\cos x}{-\sin x}=0.$$

Ex. 2. Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

[*Patna* 06, *Delhi* 07, *Kumaon* 10, *Allahabad* 10, *Kanpur* 11]

We have $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ $\qquad(\infty-\infty)$

$$=\lim_{x\to 0}\left(\frac{\sin^2 x-x^2}{x^2\sin^2 x}\right)\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin^2 x-x^2}{x^4}\right).\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2$$

$$\left(\because \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\right)$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\sin 2x-2x}{4x^3}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2\cos 2x-2}{12x^2}=-\frac{1}{6}\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

$$=-\frac{1}{6}\lim_{x\to 0}\frac{2\sin^2 x}{x^2}=-\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=-\frac{1}{3}.$$

16.4. The forms 0^0, 1^∞, ∞^0. These can be made to depend upon one of the previous forms.

Let $\qquad A=\lim_{x\to a}\{\phi(x)\}^{\psi(x)}$,

then $\qquad \log A=\lim_{x\to a}\{\psi(x)\log\phi(x)\}$,

on taking the logarithm of both sides.

Thus if $\{\phi(x)\}^{\psi(x)}$ assumes any of the forms 0^0, 1^∞ or ∞^0, we have only to determine the limit of $\psi(x)\log\phi(x)$ by one of the previous methods.

The student sometimes wonders why 1^∞ should be an indeterminate form, seeing that 1 raised to any power gives 1. But he should notice that this only entitles him to conclude that if

$$\lim_{x\to a}\psi(x)=\infty,\qquad\text{then}\qquad\lim_{x\to a}1^{\psi(x)}=1.$$

When $\phi(x)\to 1$ as $x\to a$ and at the same time $\psi(x)\to\infty$ as $x\to a$, then $\lim\{\phi(x)\}^{\psi(x)}$ might be anything; for the value of $\{\phi(x)\}^{\psi(x)}$ when $\phi(x)$ is not exactly, but only approximately, equal to 1, is not necessarily near to 1 when $\psi(x)$ is large. Consider for example, the value of $(1.01)^{1000000}$. Applying the Binomial Theorem, we see at once that it is far greater than 1.

Ex. 1. Evaluate $\lim_{x\to\infty}(1+a/x)^x$.

Let $\qquad A=\lim_{x\to\infty}(1+a/x)^x.\qquad(1^\infty)$.

Then $\qquad \log A=\lim_{x\to\infty}\{x\log(1+a/x)\}\qquad(0\times\infty)$

$$=\lim_{x\to\infty}\frac{\log(1+a/x)}{1/x}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\infty}\frac{(1+a/x)^{-1}(-ax^{-2})}{-x^{-2}}$$

$$=\lim_{x\to\infty}a(1+a/x)^{-1}=a.$$

Hence $\quad A=e^a$.

Ex. 2. Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$

[*Garhwal* 03, *Patna* 03, 05, 09, *Rohilkhand* 10, *Magadh* 02, 05, 06, 11]

Let $\qquad A=\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$ $\qquad$ (which is of the form 1^∞)

Then $\qquad \log A=\lim_{x\to 0}\frac{1}{x}\cdot\log\frac{\tan x}{x},\qquad(0\times\infty)$

$$=\lim_{x\to 0}\left[\frac{1}{x}\log\left\{\frac{x+x^3/3+2x^5/15+\dots}{x}\right\}\right]$$

$$=\lim_{x\to 0}\left[\frac{1}{x}\log\{1+(\tfrac{1}{3}x^2+\tfrac{2}{15}x^4+\dots)\}\right]$$

$$=\lim_{x\to 0}\left[\frac{1}{x}\left\{(\tfrac{1}{3}x^2+\tfrac{2}{15}x^4+\dots)-\tfrac{1}{2}(\tfrac{1}{3}x^2+\tfrac{2}{15}x^4+\dots)^2+\dots\right\}\right]$$

$$=\lim_{x\to 0}\left[\frac{1}{x}(\tfrac{1}{3}x^2+\tfrac{7}{90}x^4+\text{terms with higher powers of }x)\right]$$

$$=\lim_{x\to 0}(\tfrac{1}{3}x+\tfrac{7}{90}x^3+\text{terms with higher powers of }x)=0.$$

Hence $\quad A=e^0=1$.

Ex. 3. Evaluate $\lim_{x\to\pi/2}(\sin x)^{\tan x}$.

[*Lucknow* 05, *Jabalpur* 05, *Calcutta* 05, *Magadh* 07]

Let $A=\lim_{x\to\pi/2}(\sin x)^{\tan x}$ (which is of the form 1^{∞})

Then $\log A=\lim_{x\to\pi/2}\tan x\log\sin x=\lim_{x\to\pi/2}\dfrac{\log\sin x}{\cot x}$ $\left(\frac{0}{0}\right)$

$\frac{1}{\cot x}$

$$=\lim_{x\to\pi/2}\frac{\cos x/\sin x}{-\operatorname{cosec}^2 x}=\lim_{x\to\pi/2}(-\sin x\cos x)=0.$$

Hence $A=e^0=1$.

Ex. 4. Evaluate $\lim_{x\to0}(\cot x)^x$. [*Himachal* 2006]

Let $A=\lim_{x\to0}(\cot x)^x$. (which is of the form ∞^0)

Then $\log A=\lim_{x\to0}x\log\cot x=\lim_{x\to0}x\cdot(\log\cos x-\log\sin x)$

$$=\lim_{x\to0}x\log\cos x-\lim_{x\to0}x\log\sin x$$

$$=0-\lim_{x\to0}\frac{\log\sin x}{1/x}\quad\left(\frac{\infty}{\infty}\right)$$

$$=-\lim_{x\to0}\frac{\cos x/\sin x}{-1/x^2}=\lim_{x\to0}\frac{x^2}{\tan x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\frac{2x}{\sec^2 x}=2\lim_{x\to0}x\cos^2 x=0.$$

Hence $A=e^0=1$.

16.5. Compound forms. In case a given function can be broken up into two or more factors, the limit of each of which can be easily found (either by the methods of this chapter or by mere substitution), then the limit of the entire function can be determined by evaluating the limit of each factor separately, provided that the product of these limits is not itself an indeterminate form.

A similar rule would apply in the case of a quotient, sum, difference or power. The student should be on the look out for recognising cases in which such methods are applicable.

Ex. Evaluate $\lim_{x\to0}\left\{\left(\dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}\right)^2+\dfrac{1-\cos x}{\cos x\sin^2 x}\right\}$.

[*Allahabad* 2006]

Let $\lim_{x\to0}\left\{\left(\dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}\right)^2+\dfrac{1-\cos x}{\cos x\sin^2 x}\right\}=A^2+B$ (1)

where $A=\lim_{x\to0}\dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}$ and $B=\lim_{x\to0}\dfrac{1-\cos x}{\cos x\sin^2 x}$

Now $A=\lim_{x\to0}\dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}$ $\left(\frac{0}{0}\right)$

$$=\lim_{x\to0}\frac{\sin 2x+2\sin^2 x-2\sin x}{1-\cos x}\cdot\frac{1}{\cos x}$$

$$=\lim_{x\to0}\frac{\sin 2x+2\sin^2 x-2\sin x}{1-\cos x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\frac{2\cos 2x+4\sin x\cos x-2\cos x}{\sin x}$$

$$=\lim_{x\to0}\frac{2\cos 2x+2\sin 2x-2\cos x}{\sin x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\frac{-4\sin 2x+4\cos 2x+2\sin x}{\cos x}=4.\qquad(2)$$

Also $B=\lim_{x\to0}\dfrac{1-\cos x}{\cos x\sin^2 x}=\lim_{x\to0}\dfrac{1-\cos x}{\cos x\,(1-\cos^2 x)}$

$$=\lim_{x\to0}\frac{1}{\cos x\,(1+\cos x)}=\frac{1}{2}\qquad(3)$$

Substituting (2) and (3) in (1), the required limit

$$=A^2+B=4^2+\tfrac{1}{2}=16\tfrac{1}{2}.$$ 33/2

EXAMPLES

Evaluate the following limits:

1. $\lim_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$ [*Indore* 2008]

2. $\lim_{x\to1}\left(\dfrac{2}{x^2-1}-\dfrac{1}{x-1}\right)$ [*Agra* 08, 10]

3. $\lim_{x\to\frac{1}{2}\pi}\left(x\tan x-\tfrac{1}{2}\pi\sec x\right)$.

4. $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\tan x}\right)$. [*Kumaon* 2011]

5. $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$. [*Delhi* 05, *Kumaon* 09]

6. $\lim_{x\to\frac{1}{2}\pi}\left(\sec x-\frac{1}{1-\sin x}\right)$. [*Agra* 2011]

7. $\lim_{x\to 0}(a^x+x)^{1/x}$. [*Allahabad* 2004]

8. $\lim_{x\to 0}(1/x)^{\tan x}$. [*Magadh* 2010]

9. $\lim_{x\to 0}(\cos x)^{1/x^2}$.

10. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$. [*Kumaon* 03, *Allahabad* 05, *Himachal* 05]

11. $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$. [*Purvanchal* 05, *Rohilkhand* 11]

12. $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^3}$.

13. $\lim_{x\to 0}(\cot x)^{\sin x}$. [*Himachal* 2007]

14. $\lim_{x\to 0}(\cos x)^{\cot x}$. [*Avadh* 2003]

15. $\lim_{x\to 0}(\cos x)^{\cot^2 x}$. [*Lucknow* 11, 12]

16. $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan(\pi x/2a)}$. [*Avadh* 04, *Patna* 11]

17. $\lim_{x\to 1} x^{1/(x-1)}$. [*Allahabad* 2011]

18. $\lim_{x\to 1}(1-x^2)^{1/\log(1-x)}$.

19. $\lim_{x\to e}(\log x)^{1/(1-\log x)}$.

20. $\lim_{x\to a}(x-a)^{x-a}$.

21. $\lim_{x\to\infty}\left(\frac{1}{2}\pi-\tan^{-1}x\right)^{1/x}$. [*Jabalpur* 2008]

ANSWERS

1. 0. **2.** $-\frac{1}{2}$. **3.** -1. **4.** $\frac{1}{3}$.
5. $\frac{2}{3}$. **6.** $-\infty$. **7.** ae. **8.** 0.
9. $e^{-1/2}$. **10.** $e^{-1/6}$. **11.** $e^{1/3}$.
12. ∞ on the right, 0 on the left. **13.** 1. **14.** 1.
15. $e^{-1/2}$. **16.** $e^{2/\pi}$. **17.** e. **18.** e.
19. $1/e$. **20.** 1 $(x>a)$. **21.** 1.

Notes